MECHANICS OF FLUIDS.

CHURCH.

HYDRAULICS;

OR

MECHANICS

OF

FLUIDS.

Being Part IV of the MECHANICS OF ENGINEERING *by Irving P. Church, C.E.*

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391. A PERFECT FLUID is a substance the particles of which are capable of moving upon each other with the greatest freedom, absolutely without friction, and are destitute of mutual attraction. In other words, the stress between any two contiguous portions of a perfect fluid is always one of compression and normal to the dividing surface at every point; i.e., NO SHEAR or tangential action can exist on any imaginary cutting plane.

Hence if a perfect fluid is contained in a vessel of rigid material the pressure experienced by the walls of the vessel is normal to *the surface of contact at all points.*

For the practical purposes of Engineering, water, alcohol, mercury, air, steam and all gases may be treated as perfect fluids at ~~ordinary~~ certain temperatures.

392. LIQUIDS AND GASES. A fluid a definite mass of which occupies a definite volume at a given temperature, and is incapable both of expanding into a larger volume and of being compressed into a smaller volume, is called a LIQUID, of which water, mercury etc. are common examples; whereas a GAS is a fluid a mass of which is capable of almost indefinite expansion or compression, according as the space within the confining vessel is made larger or smaller, and always tends to fill the vessel which must ∴ be closed in every direction to prevent its escape.

Liquids are sometimes called *inelastic* fluids, and gases *elastic* fluids.

393. REMARKS. Though practically we may treat all liquids as incompressible, experiment shows them to be compressi-

ible to a slight extent. Thus, a cubic inch of water under a pressure of 15 lbs. on each of its six faces loses only fifty millionths (0.000050) of its original volume. The slight cohesion existing between the particles of most liquids is too insignificant to be considered in the present connection.

The property of indefinite, on the part of gases, by which a confined mass of gas can continue to fill a confined space which is progressively enlarging, and exert pressure against its walls, is satisfactorily explained by the "Kinetic Theory of Gases", according to which the gaseous particles are perfectly elastic and in continual motion, impinging against each other and the confining walls. Nevertheless, for practical purposes, we may consider a gas as a continuous substance.

Although by the abstraction of heat, or the application of great pressure, or both, all known gases may be reduced to liquids, (some being even solidified); and although by converse processes (imparting heat and diminishing the pressure) liquids may be transformed into gases, the range of temperature and pressure in all problems to be considered in this work is supposed kept within such limits that no extreme changes of state, of this character, take place. A gas approaching the point of liquefaction is called a VAPOR.

Between the solid and the liquid state we find all grades of intermediate conditions of matter. For example, some substances are described as soft and plastic solids, as soft putty, moist earth, pitch, and fresh mortar; and others as viscous and sluggish liquids, as molasses, glycerine. Such as these are not considered in the present treatise.

394. HEAVINESS OF FLUIDS. The weight of a cubic unit of a homogeneous fluid will be called its *heaviness*, or rate of weight, (see § 7) and is a measure of its density. Denoting it by γ (*gamma*) and the volume of a definite portion of the fluid by V, we have, for the weight

of that portion, $G = V\gamma$

This, like the great majority of equations to be used or derived in this work, is of homogeneous form (§ 6) i.e., admits of any system of units. E.g., in the metre-kilogram-second system, if γ is given in kilos. per cubic metre, V must be expressed in cubic metres and G will be obtained in kilos.; and similarly in any other system. The quality of $\gamma = G \div V$ is evidently one dimension of force divided by three dimensions of length.

In the following table, in the case of gases the temperature and pressure are mentioned at which they have the given heaviness, since under other conditions the heaviness would be different; in the case of liquids, however, for ordinary purposes the effect of a change of temperature may be neglected.

HEAVINESS OF VARIOUS FLUIDS [in ft. lb. sec. sys.]

γ = weight in lbs. of a cubic foot.

Fresh water $\gamma =$	62.5	Gases { At temp. of freezing; and 14.7 lbs. per sq. in tension	
Sea " "	64.0		
Mercury	848.7		
Alcohol	49.3	Atmospheric Air	0.08076
Crude Petroleum about	55.0	Oxygen	0.0892
N.B. A cubic inch of water		Nitrogen	0.0786
weighs 0.036024 lbs.;		Hydrogen	0.0056
and a cubic foot 1000 av. oz.		Illuminating Gas } from	0.0300
		} to	0.0400

Example 1. What is the heaviness of a gas, 432 cub. in. of which weigh 0.368 ounces? Use ft. lb. sec. system.

432 cub. in. = $\frac{1}{4}$ cub. ft. and 0.368 oz. = 0.023 lbs.

$$\therefore \gamma = \frac{G}{V} = \frac{0.023}{1/4} = 0.092 \text{ lbs. per cub. foot.}$$

Example 2. Required the weight of a right prism of mercury one sq. inch section and 30 inches altitude.

$V = 30 \times 1 = 30$ cub. in. $= \frac{30}{1728}$ cub. feet, while

from the table, γ for mercury $= 848.7$ lbs. per cub. ft.

$\therefore$ its weight $= G = V\gamma = \frac{30}{1728} \times 848.7 = 14.73$ lbs.

395. DEFINITIONS. By *Hydraulics* (called also Hydromechanics by some recent writers) we understand the mechanics of fluids as utilized in Engineering. It may be divided into

Hydrostatics, treating of fluids at rest; and

Hydrodynamics (or Hydrokinetics) which deals with fluids in motion. (The name Pneumatics is sometimes used to cover both the statics and dynamics of gaseous fluids)

Before treating separately of liquids and gases, a few paragraphs will be presented applicable to both kinds of fluid, at rest.

396. PRESSURE PER UNIT AREA, or INTENSITY OF PRESSURE. As in § 180 in dealing with solids, so here with fluids, we indicate the pressure per unit area between two contiguous portions of fluid, or between a fluid and the wall of the containing vessel by p, so that if dP is the total pressure on a small area dF, we have

$$p = \frac{dP}{dF} \quad \ldots\ldots\ldots (1)$$

as the pressure per unit area, or intensity of pressure (often called the *tension* in speaking of a gas) on the small surface dF. If pressure of the <u>same</u> <u>intensity</u> exists over a finite plane surface of area $= F$, the total pressure on that surface is

$$\left.\begin{aligned} P = \int p\,dF = p\int dF = Fp \\ \text{or } p = \frac{P}{F} \end{aligned}\right\} \quad \ldots\ldots (2)$$

(N.B. For brevity the single word pressure will sometimes be used, instead of intensity of pressure, where no ambiguity would arise). Thus, it is found that under ~~ordinary~~ conditions at the sea level the atmosphere exerts a normal pressure (normal, because fluid pressure) on all surfaces, of an intensity of about $p = 14.7$ lbs. per sq. inch ($= 2116.$ lbs per sq. ft.,

and called *one atmosphere*) so that the total atmospheric pressure on a surface of 100 sq. in., for example, is (in. lb. sec)

$$P = Fp = 100 \times 14.7 = 1470 \text{ lbs.} \quad (= 0.735 \text{ tons})$$

The quality of p is evidently one dimension of force over two dimensions of length

397. HYDROSTATIC PRESSURE; PER UNIT AREA, in the interior of a fluid at rest. In a body of fluid of uniform heaviness, at rest, it is required to find the pressure per unit area between the portions of fluid on opposite sides of any imaginary cutting plane. As customary, we shall consider portions of the fluid as free bodies, by supplying the forces exerted on them by all contiguous portions (of fluid or vessel wall) also those of the earth (their weights), and then apply the conditions of equilibrium. *First, cutting plane horizontal.* Fig. 443 shows a body of homogeneous fluid confined in a rigid vessel closed at the top with a small air-tight but frictionless piston, (a horizontal disc) of weight $= G$ and exposed to atmospheric pressure ($= p_a$ per unit area) on its upper face. Let the area of piston-face be $= F$. Then for the equilibrium of the piston the pressure between its under surface and the fluid at O must be (total) $P = G + Fp_a$ and the intensity of this pressure is

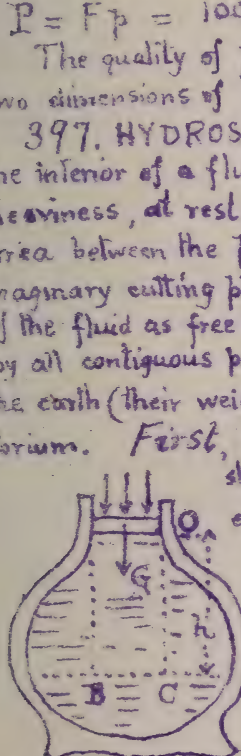


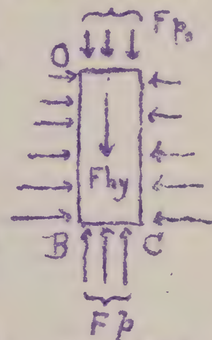
Fig. 443

$$\therefore \quad p_o = \frac{G}{F} + p_a \quad \ldots\ldots (1)$$

It is now required to find the intensity, p, of fluid pressure between the portions of fluid contiguous to horizontal cutting plane BC at a vertical distance $= h$ vertically below the piston O. In Fig. 444 we have as a free body the right parallelopiped OBC of Fig. 443 with vertical sides (two ∥ to paper and four ⊥ to it). The pressures acting on its six faces are normal to them respective-

h, and the weight of the prism is $=$ vol. $\times \gamma = Fh\gamma$, supposing γ to have the same value at all parts of the column [This is practically true for any height of liquid and for a small height of gas] Since the prism is in equil. under the forces shown in the figure and would still be so were it to become rigid we may put (§ 36) Σ(vert. compons.) $= 0$ and $\therefore$ obtain

$$Fp - Fp_0 - Fh\gamma = 0 \; \ldots\ldots (2)$$



(In the figure the pressures on the vertical faces ∥ to paper have no vert. compons. hence are not drawn) From (2) we have

$$p = p_0 + h\gamma \; \ldots\ldots (3)$$

Fig. 444

($h\gamma$ being the weight of a column of homogeneous fluid of unity cross-section and height h, would be the total pressure on the base of such a column, if at rest with no pressure on upper base, and hence might be called *intensity due to weight.*)

Secondly, cutting plane oblique. Fig. 445. Consider free an infinitely small right triangular prism bcd, whose bases are ⊥ to the paper, while the three side faces (rectangles), having areas $= dF$, dF_1, and dF_2, are respectively horizontal, vertical and oblique; let angle $bcd = \alpha$. The surface bc is a portion of the plane BC of ~~Fig. 444~~. Given p (= pressure on dF) and α, required p_2 the intensity of pressure on the oblique face bd, of area dF_2.

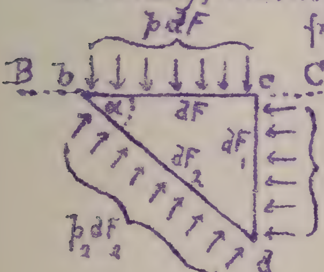


Fig. 445

[N.B. The prism is taken very small in order that the intensity of pressure may be considered constant over any one face; and also that the weight of the prism may be neglected, since it involves the volume (three dimensions) of the prism while the total face pressures involve only two, and is $\therefore$ a differential of a higher order]

From Σ (vert. compons.) $= 0$ we shall have

$p_2\, dF_2 \cos\alpha - p\,dF = 0$; but $dF \div dF_2 = \cos\alpha$

$\therefore \quad p_2 = p$ . . . . (4)

which is independent of the angle α.

Hence, the intensity of fluid pressure at a given point is the same on all imaginary cutting planes containing the point. This is the most important property of a fluid.

398 The **INTENSITY OF PRESSURE IS EQUAL AT ALL POINTS OF ANY HORIZONTAL PLANE** in a body of homogeneous fluid at rest. If we consider a right prism of the fluid in Fig 443, of small vertical thickness, its axis lying in any horizontal plane BC, its bases will be vertical and of equal area dF. The pressures on its sides, being normal to them, and hence to the axis, have no components ∥ to the axis. The weight of the prism also has no horizontal component. Hence from Σ(hor. comps.) = 0, we have, p_1 and p_3 being the press.-intensities at the two bases, $p_1 dF - p_3 dF = 0 \quad \therefore p = p_3$ . . . . . (1)

which proves the statement at the head of this article.

It is now plain, from this and the preceding article, that the press.-intensity p at any point in a homogeneous fluid at rest is *equal to that at any higher point, plus the weight ($h\gamma$) of a column of the fluid of section unity and of altitude (h) = vertical distance between the points*

i.e. $\quad p = p_0 + h\gamma$ . . . . (2)

whether they are in the same vertical or not, and whatever be the shape of the containing vessel (or pipes), provided the fluid is continuous between the two points; for, Fig. 446, by considering a series of small prisms, alternately vertical and horizontal, o b c d e, we know that

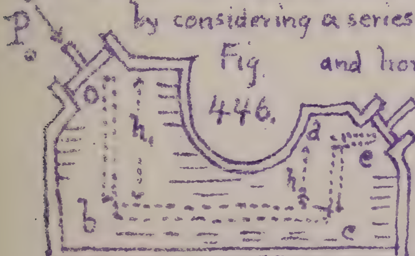


Fig. 446.

$p_b = p_o + h_1\gamma$, $p_c = p_b$

$p_d = p_c - h_2\gamma$, and $p_e = p_d$;

hence, finally, by addition we have

$p_e = p_0 + h\gamma$, (in which $h = h_1 - h_2$)

If ∴, upon a small piston at O, of area $= F_0$, a force P_0 be exerted, and an inelastic fluid (liquid) completely fills the vessel, then, for equilibrium, the force to be exerted upon the piston at e, viz. P_e, is thus computed: For equilibrium of fluid $p_e = p_0 + h\gamma$ and for equil. of piston O, $p_0 = P_0 \div F_0$; also $p_e = P_e \div F_e$.

$$\therefore\ P_e = \frac{F_e}{F_0} P_0 + F_e h\gamma \quad \ldots \quad (3)$$

From (3) we learn that if the pistons are at the same level ($h=0$) the total pressures on their inner faces are *directly proportional to their areas.*

If the fluid is gaseous (2) and (3) are practically correct if h is not $>$ 100 feet, (for being compressible the lower strata are generally more dense than the upper), but in (3) the pistons must be fixed and P_e and P_0 refer solely to the interior pressures.

Again, if h is small or p_0 very great, the term $h\gamma$ may be omitted altogether in eqs (2) and (3), (especially with gases, since for them γ (heaviness) is usually small) and we then have (2)

$$p = p_0 \quad \ldots \quad (4)$$

being the algebraic form of the statement: *A body of fluid at rest transmits pressure with equal intensity in every direction and to all of its parts.* [Principle of "Equal Transmission of Pressure"]

399. MOVING PISTONS. If the fluid in Fig. 446 (c) is inelastic, and the vessel walls rigid, the motion of one piston through a distance s_0 causes the other to move through a distance s_e determined by the relation $F_0 s_0 = F_e s_e$ (since the volumes described by them must be equal) but on account of the inertia of the liquid, and friction on the vessel walls, equations (2) and (3) no longer hold exactly, but are approximately true if the motion is very slow and the vessel short, as with the cylinder of a water-pressure engine.

But if the fluid is compressible and elastic (gases and vapors; steam, or air) and hence of small density, the effect of inertia.

and friction is not appreciable in short wide vessels like the cylinders of steam- and air-engines, and those of air compressors; and eq.s (2) and (3) still hold, practically even with high piston-speeds. For example, in the space AB, Fig. 447, between the piston and cylinder-head of a steam engine (piston moving toward the right) the intensity of pressure, p, of the steam against the moving piston B is practically equal to that against the cylinder-head A at the same instant

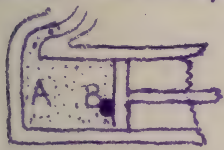


Fig. 447

399. AN IMPORTANT DISTINCTION between gases and liquids (i.e. between elastic and inelastic fluids) consists in this:

A liquid can exert pressure against the walls of the containing vessel only by its weight, or (when confined on all sides) by transmitting pressure coming from without (due to piston pressure, atmospheric pressure, etc.); whereas

A gas, confined, as it must be, on all sides to prevent diffusion, exerts pressure on the vessel not only by its weight but but by its elasticity or tendency to expand. If pressure from without is also applied, the gas is compressed and exerts a still greater pressure on the vessel walls.

400. COMPONENT, OF PRESSURE, IN A GIVEN DIRECTION. Fig 448. Let ABCD, whose area $= dF$, be a small element of a surface, plane or curved, and p the intensity of fluid pressure upon this element, then the total pressure upon it is $p dF$ and is of course normal to it. Let $A'B'CD$ be the projection of the element dF upon a plane CDM making an angle α with the element,

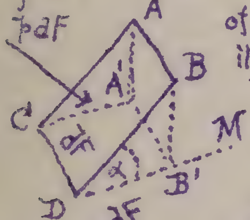


Fig. 448.

and let it be required to find the value of the component of pdF in a direction *normal* to this last plane (the other component being ∥ to the same plane). We shall have

Compon. of pdF ⊤ to CDM is $pdF\cos\alpha = p(dF\cos\alpha)$.....(1)

But $dF\cos\alpha =$ area $A'B'CD$, the projection of dF upon the plane

CDM ∴ Compon. $\perp$ to plane CDM $= p \times$ (projec. of dF on CDM)
i.e. the component of fluid pressure (on an element of the surface) *in a given direction* (the other component being $\perp$ to the first) *is found by multiplying the intensity of the pressure by the area of the projection of the element upon a plane $\perp$ to the given direction.*

401. NON-PLANAR PISTONS. From the foregoing it follows that the sum of the components $\parallel$ to the piston-rod, of the fluid pressures upon the piston at (A) Fig. 449 is just the same as at (B), if the cylinders are of equal size and the steam or air is at the same tension. For the sum of the projections of all the elements of the curved surface of A upon a plane $\perp$ to the piston-rod is always $= \pi r^2 =$ area of section of cylinder-bore. If the surface of A is symmetrical about the axis of the cylinder the other components (i.e. those $\perp$ to the piston-rod) will neutralize each other. If that surface is irregular, however, the piston may be pressed laterally against the cylinder-wall but the thrust along the rod or "*working force*" (§ 128) is the same, in all instances, as if the surface were plane and $\perp$ to piston rod.

Fig. 449

402. BRAMAH, OR HYDRAULIC, PRESS. This is a familiar instance of the principle of ~~transmis~~-sion of fluid pressure. Fig. 456. Let the small piston at O have a diameter $d =$ 1 inch $= \frac{1}{12}$ ft., while the plunger E, or large piston has a diameter $d' = AB = CD = 15$ in. $= \frac{5}{4}$ ft. The lever MN weighs $G_1 = 3$ lbs., and a weight $G = 40$ lbs. is hung at M. The lever-arms of these forces about the fulcrum N are given in the figure. The apparatus being full of

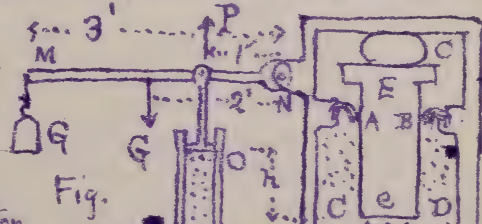


Fig. 456.

water, the ~~liquid~~ pressure P against the small piston is found by putting Σ(moms. about N) = 0 for the equilibrium of the lever; whence [ft, lb. sec.]

$$P \times 1 - 40 \times 3 - 3 \times 2 = 0 \quad \therefore P = 126 \text{ lbs.}$$

But, denoting atmos. pressure by p_a and that of the water against the piston by p_o (per unit area), we may also write

$$P = F_o p_o - F_o p_a = \tfrac{1}{4}\pi d^2 (p_o - p_a) \qquad \text{Solving for } p_o$$

we have, putting $p_a = 14.7 \times 144$ lbs. per sq. ft.,

$$p_o = \left[126 \div \frac{\pi}{4}\left(\frac{1}{12}\right)^2\right] + 14.7 \times 144 = 25236. \text{ lbs per sq. ft}$$

Hence at e the press. per unit area, from (2) § 398, and § 394 is

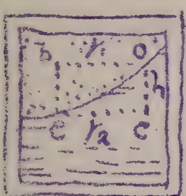
$$p_e = p_o + h\gamma = 25236. + 3 \times 62.5 = 25423. \text{ lbs per sq. ft.}$$

$= 175.6$ lbs. per sq. inch or 11.9 atmospheres, and the total upward pressure at e on base of plunger is

$$P = F_e p_e = \pi \frac{d'^2}{4} p_e = \frac{1}{4}\pi\left(\frac{5}{4}\right)^2 25423 = 31194 \text{ lbs.}$$

or almost 16 tons (of 2000 lbs. each). The compressive force upon the block or bale, C, will = P less the weight of the plunger and total atmos. pressure on a circle of 16 in. diameter

403. THE DIVIDING SURFACE OF TWO FLUIDS (WHICH DO NOT MIX) IN CONTACT, AND AT REST, IS A HORIZONTAL PLANE. For, Fig. 457, supposing any two points e and o of this surface to be at different levels (the pressure at o being p_o, that at e p_e, and the heavinesses of the two fluids γ_1 and γ_2 respectively) we would have, from a consideration of the two elementary prisms eb and bo (vertical and horizontal) the relation



$p_e = p_o + h\gamma_1$, while from the prisms ec and co, the relation $p_e = p_o + h\gamma_2$. These equations are conflicting hence the above supposition is absurd. Hence the proposition is true.

For stable equilibrium, evidently, the heavier fluid must occu-

by the lowest position in the vessel, and if there are several fluids (which do not mix), they will arrange themselves vertically, in the order of their densities, the heaviest at the bottom, Fig. 458. On account of the property called *diffusion* the particles of two gases placed in contact soon intermingle and form a uniform mixture. This fact gives strong support to the "Kinetic Theory of Gases." (§ 393).

Fig. 458

404. FREE SURFACE OF A LIQUID AT REST. The surface (of a liquid) not in contact with the walls of the containing vessel is called a free surface, and is necessarily horizontal (from § 403) when the liquid is at rest. Fig. 459. [A gas from its tendency to indefinite expansion is incapable of having a free surface.]

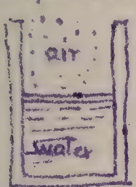
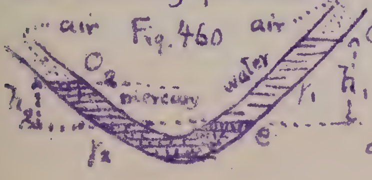


Fig. 459

This is true even if the space above the liquid is vacuous, for if the surface were inclined or curved, points in the body of the liquid and in the same horizontal plane, would have different heights (or "heads") of liquid between them and the surface, producing different intensities of pressure in the plane, which is contrary to § 398.

When large bodies of liquid, like the ocean, are considered, gravity can no longer be regarded as acting in parallel lines; consequently the free surface of the liquid is curved being ⊥ to the direction of (apparent) gravity at all points. For ordinary engineering purposes (except in Geodesy) the free surface of water at rest is practically horizontal.

405. TWO LIQUIDS (which do not mix) AT REST IN A BENT TUBE OPEN AT BOTH ENDS TO THE AIR, Fig. 460; water and mercury for instance. Let their heavinesses be γ_1 and γ_2 respectively. The pressure at e may be written (§ 398)



$$p_e = p_{o_1} + h_1\gamma_1 \quad \ldots\ldots (1)$$

or

$$p_e = p_{o_2} + h_2\gamma_2 \quad \ldots\ldots (2)$$

according as we refer it to the water column or the mercury column and their respective free surfaces where the pressure $p_{o_1} = p_{o_2} = p_a$ = atmos. press. e is the surface of contact of the two liquids. Hence we have

$$p_a + h_1\gamma_1 = p_a + h_2\gamma_2 \quad \text{i.e., } h_1 : h_2 :: \gamma_2 : \gamma_1 \quad \ldots\ldots (5)$$

i.e. *the heights of the free surfaces of the two liquids above the surface of contact are inversely proportional to their respective heavinesses.*

Example. If the pressure at e = 2 atmospheres (§ 396) we shall have from (1) (inch-lb.-sec. system of units)

$$h_1\gamma_1 = p_e - p_a = 2 \times 14.7 - 14.7 = 29.4 \text{ lbs. per sq. inch.}$$

$$\therefore h_1 \text{ must} = 29.4 \div [848.7 \div 1728] = 30 \text{ inches.}$$

(since for mercury γ_1 = 848.7 lbs. per cub. ft.) Hence from (3)

$$h_2 = \frac{h_1\gamma_1}{\gamma_2} = \frac{30 \times [848.7 \div 1728]}{62.5 \div 1728} = 408^{\text{inches}} = 34 \text{ feet}$$

I.e., for equilibrium and that p_e may = 2 atmospheres, h_1 and h_2 (of water and mercury) must 30 in. and 34 feet respectively.

406. CITY WATER PIPES. If h = vertical distance of a point B of a water pipe below the free surface of a reservoir, and the water be at *rest*, the pressure on the inner surface of the pipe at B (per unit of area) is

$$p = p_o + h\gamma \qquad \text{and here } p_o = p_a = \text{atmos. press.}$$

Example. If h = 200 ft. (using the inch, lb., and second)

$$p = 14.7 + [200 \times 12][62.5 \div 1728] = 101.5 \text{ lbs. per sq. in.}$$

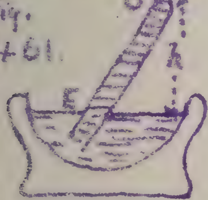
The term $h\gamma$, alone, = 86.8 lbs. per sq. inch, is spoken of as the *hydrostatic pressure* due to 200 feet height, or "HEAD", of water. (See *Trautwine's Pocket-Book* for a table of hydrostatic pressures for various depths.)

If, however, the water is *flowing* through the pipe, the pressure against the interior wall becomes, (a problem of Hydrodynamics

to be treated subsequently) while if that motion is suddenly checked the pressure becomes momentarily much greater than the hydrostatic. This shock is called "water-ram", and "water-hammer".

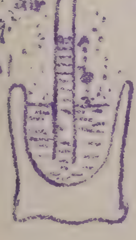
407. BAROMETERS AND MANOMETERS FOR FLUID PRESSURE. If a tube, closed at one end, be filled with water, and the other extremity, temporarily stopped, is afterwards opened under water, the closed end being then a (vertical) height $= h$, above the surface of the water, it is required to find the intensity of fluid pressure at the top of the tube, p_0, supposing it to remain filled with water. Fig. 461. At E inside the tube the pressure is 14.7 lbs. per sq. inch, the same as that outside at the same level (§ 398), hence, from $p_E = p_0 + h\gamma$, $p_0 = p_E - h\gamma$ ......(1)

Fig. 461.



Example. Let $h = 10$ feet, (with inch, lb., sec. system); then $p_0 = 14.7 - 120 \times [62.5 \div 1728] = 10.4$ lbs. per sq. inch, or about $\frac{2}{3}$ of an atmosphere. If now we inquire the value of h to make $p_0 =$ zero, we put $p_E - h\gamma = 0$ and obtain $h = 408$ inches $= 34$, which is called the height of *the water barometer*. Hence, Fig. 461, ordinary atmospheric pressure will not sustain a column of water higher than 34 feet. If mercury is used instead of water the height supported by one atmosphere will be $b = 14.7 \div [848.7 \div 1728] = 30$ inches $= 76$ centims (about), ~~and the tube~~ is of more manageable proportions than with water, besides the advantage that no vapor of mercury forms above the liquid at ordinary temperatures [In fact the water barometer height $b = 34$ feet has only a theoretical existence since at ordinary temperatures (40 to 80° Fahr) vapor of water would form above the column and depress it by from 0.30 to 1.09 ft.] Such an apparatus is called a

Fig. 461



Barometer and is used not only for measuring the varying tension of the atmosphere (from 14.5 to 15 lbs. per sq. inch) but also that of any body of gas. Thus, Fig. 462, the gas in D is put in communication with the space above the mercury in the cistern at C; and we have $p = h\gamma$, where γ = heav. of mercury. For delicate measurements an attached thermometer is also used, as the heaviness γ varies slightly with the temperature.

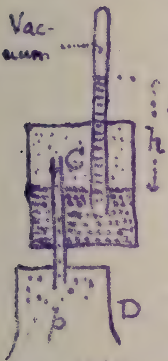


Fig. 462

If the vertical distance CD is small, the tension in C is considered the same as in D.

For gas-tensions greater than one atmosphere, the tube may be left open at the top, forming an OPEN MANOMETER, Fig. 463. In this case, the tension of the gas above the mercury in the cistern is

$$p = (h + b)\gamma \quad \ldots\ldots\ldots\ldots (1)$$

in which b is the height of mercury (about 30 in.) to which the tension of the atmosphere above the mer. column is equivalent.

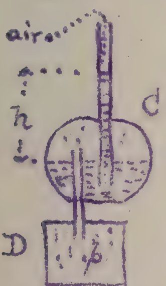


Fig. 463

Example. If $h = 51$ inches, Fig. 463, we have (ft. lb. sec.)

$$p = [4.25^{ft} + 2.5^{ft}]\,848.7 = 5728 \text{ lbs. per sq. foot.}$$

$$= 39.7 \text{ lbs. per sq. inch} = 2.7 \text{ atmospheres.}$$

Another form of the open manometer consists of a U tube, Fig. 464, the atmosphere having access to one branch, the gas to be examined, to the other, while the mercury lies in the curve. As before, we have

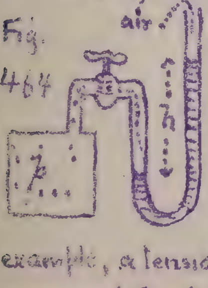


Fig. 464

$$p = (h + b)\gamma = h\gamma + p_a \quad \ldots\ldots (2)$$

where p_a = atmos. tension, and b as above.

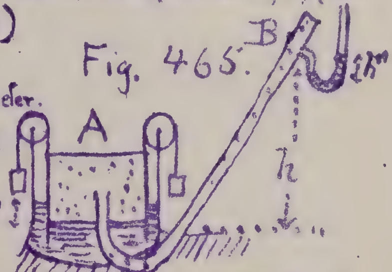
The tension of a gas is sometimes spoken of as measured by so many inches of mercury. For example, a tension of 22.05 lbs. per sq. inch ($1\frac{1}{2}$ atmos.) is measured by 45 inches of mercury in a vacuum manometer, (i.e., a common barometer) Fig. 462. With the open manometer

this tension ($1\frac{1}{2}$ atmos.) would be indicated by 15 inches of actual mercury, Figs 463 and 464. An ordinary steam gauge indicates the excess of tension over one atmosphere; thus "40 lbs. of steam" implies a tension of $40 + 14.7 = 54.7$ lbs. per sq. in.

405. TENSION OF ILLUMINATING GAS. This is often spoken of as measured by *inches of water* (from 1 to 3 inches usually) Strictly it should be stated that this water-height measures the excess of its tension over one atmosphere (measured by $b = 34$ ft. $= 408$ in. of water) That is, in Fig. 464, water being used instead of mercury, $h =$ say 2 inches while $b = 408$ inches.

Example. Supposing the gas at rest, and the tension at the gasometer A, Fig. 465, to be "two inches of water", required the water column h'' (in open tube) the gas will support in the pipe at B, 120 feet (vertically) above the gasometer. Let the temperature be freezing (nearly) and the outside air at a tension of 14.7 lbs. per sq. inch. Let $\gamma_1 =$ the heaviness of atmos. (under these conditions) and $\gamma_2 =$ that of the gas, and assume that both are of constant density between A and B (which is nearly true when h (see figure) is as small as 120^{ft}). Let p_1' and p_1'' be the tensions of the outer air at A and B respectively; p_2' and p_2'' those of the gas. Then from eq (2) § 398 we have

Fig. 465.



For the water at A $\quad p_2' = p_1' + h'\gamma$ . . . . (1)

" " " " B $\quad p_2'' = p_1'' + h''\gamma$ . . . . (2)

For air between A and B, $\quad p_1' = p_1'' + h\gamma_1$ . . . . (3)

For gas " A " B, $\quad p_2' = p_2'' + h\gamma_2$ . . . . . . (4)

h'' is the chief unknown; p_2', p_1'', and p_2'' are also unknown. The known are $p_1' = 14.7$ lbs. per sq. in. (so that $p_1' \div \gamma = 408^{in.}$) $\gamma_1 = 0.0807$ lbs. per cub. foot (§ 394), γ_2 (say) $= 0.0360$ lbs. per

... feet, while γ for water $= 62.5$ lbs. per cub. ft. (Use inch lb. sec.)

From (1) we have $\dfrac{p_2'}{\gamma} = \dfrac{p_1'}{\gamma} + h' = 408 + 2 = 410$ inches

From (3) $\dfrac{p_1'}{\gamma} = \dfrac{p_1''}{\gamma} + \dfrac{h\gamma_1}{\gamma}$ $\therefore \dfrac{p_1''}{\gamma} = 408 - 120 \times 12 \times \dfrac{.0807}{62.5}$
$= 406.15$ inches of water

Similarly

from (4) $\dfrac{p_2''}{\gamma} = 410 - 120 \times 12 \times \dfrac{.0807}{62.5} = 409.17$ inches of water

Hence from (2) h'' at $B = \dfrac{p_2''}{\gamma} - \dfrac{p_1''}{\gamma} = 2.98$ say 3 inches

and is $\therefore$ greater than h' at A. Hence if a small aperture is made in the pipe at B the gas will flow out with greater velocity than at A; see Chap. VI. (For altitudes > 120 ft. see § 441).

406. SAFETY VALVES. Fig. 466. Required the proper weight G to be hung at the extremity of the horizontal lever AB, fulcrum at B, that the flat disc-valve E shall not be forced upward by the steam pressure, p', until the latter reaches a given value $= p$. Let the weight of the arm be G_1, its centre of gravity being at C, a distance $= c$ from B; the other horizontal distances are marked in the figure.

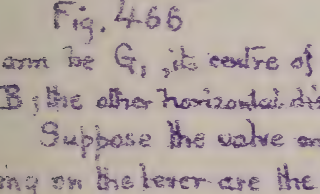


Fig. 466

Suppose the valve on the point of rising; then the forces acting on the lever are the fulcrum reaction at B, the weights G and G_1, and the two fluid-pressures on the disc viz.: Fp_a (atmospheric) downward and Fp (steam) upward. Hence from

$$\Sigma(\text{mom}_B) = 0, \quad Gb + G_1c + Fp_a a - Fpa = 0 \quad \ldots\ (1)$$

Solving, $$G = F(p - p_a) - G_1\frac{c}{a} \quad \ldots\ldots\ (2)$$

Example. With $a =$ 2 inches, $b =$ 2 feet, $c = 1$ foot;

$G_1 = 4$ lbs., $p =$ six atmospheres, and diam. of disc = 1 inch,

$$G = \frac{1}{4}\pi\left(\frac{1}{12}\right)^2\left(6 \times 14.7 \times 144 - 14.7 \times 144\right) - 4 \times \frac{1}{2} = 55.75 \text{ lbs.}$$

(Notice the canceling of the 144; for $F(p - p_a)$ is lbs. whether the inch or foot is used in both factors) Hence when the steam pressure has risen to six atmospheres (= 88.2 lbs. per sq. in., corresponding to 73.5 by steam gauge), the valve will open if $G = 55.75$ lbs.

407. PROPER THICKNESS of THIN HOLLOW CYLINDERS (i.e. pipes and tubes) to resist BURSTING BY FLUID PRESSURE.

Case I. Stresses in the cross-section due to End-pressure, Fig. 467. Let AB be the circular cap closing the end of a cylindrical tube containing fluid at a tension of p. Let r = internal radius of the tube or pipe. Then considering the cap free, neglecting its weight, we have equilibrium between the three sets

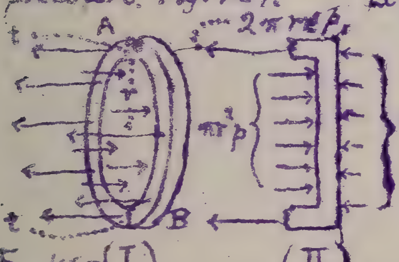


Fig. 467 (I.) (II.)

of parallel forces (see II. in Figure) viz. Internal fluid pr. $= \pi r^2 p$ External fluid press. $= \pi r^2 p_a$; while the total tensile stress on the small ring, whose area now exposed is $2\pi r t$ (nearly) is $2\pi r t p_1$, t being the thickness of the wall of pipe and p_1 the tensile stress per unit area induced by the end fluid pressure. Hence for equil.

$$\pi r^2 p - \pi r^2 p_a - 2\pi r t p_1 = 0 \quad \therefore p_1 = \frac{r(p - p_a)}{2t} \quad \dots (1)$$

Case II. Stresses in longitudinal section of pipe, due to radial fluid pressures. Fig. 468. Consider free the half (semicircle) of any length l of the pipe between two cross sections. Take an axis X as in the figure. Let p_2 = tensile stress (per unit area) produced in the straight edges (narrow rectangles) exposed at A and B (Those in the half-ring edges, having no X components, are not

in the figure). On the internal curved surface the fluid pressure is considered of equal intensity $= p$ at all points (practically true even with liquids, if $2r$ is small compared with the head of water producing p). The fluid pressure on any dF or elementary area of the internal curved surface is $= p\,dF$. Its X component (see § 400) is obtained by multiplying p by the projection of dF on the vertical plane ABC, and since p is the same for all the dF's of the curved surface, the sum of all the X components of the internal fluid pressures must $= p$ multiplied by the area of rectangle ABCD, $= 2rlp$; and similarly the X components of the external atmos. pressures $= 2rlp_a$ (nearly). The tensile stresses (∥ to X) are equal to $2lt\,p_2$; hence for equilibrium $\Sigma X = 0$ gives

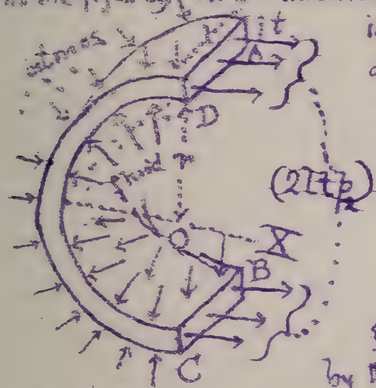


Fig. 468.

$$2ltp_2 - 2rlp + 2rlp_a = 0 \quad \therefore p_2 = \frac{r(p-p_a)}{t} \quad \ldots(2)$$

This tensile stress, called *hoop tension*, p_2, opposing rupture by longitudinal tearing, is seen to be double the tensile stress p_1 induced, under the same circumstances, on the annular cross section in Case I. Hence eq. (2) and not eq. (1) should be used to determine a safe value for the thickness of metal, t, or any other one unknown quantity involved in the equation.

For safety against rupture, we must put $p_2 = T'$, a safe tensile stress per unit area for the material of the pipe or tube; (see §§ 195 and 203)

$$\therefore t = \frac{r(p-p_a)}{T'} \quad \ldots(3)$$

Example. A pipe of twenty inches internal diameter is to contain water under a head of 340 feet; required the proper thickness, if of cast-iron.

340 feet of water measures 10 atmospheres, so that the in-

ternal fluid pressure is 11 atmospheres; but the external pressure p_a being one atmos. we must write (inch-lb.-sec.)

$(p-p_a) = 10 \times 14.7 = 147.0$ lbs per sq. in., and $r = 10$ inches, while (§ 203) we may put $T' = \frac{1}{2}$ of $9000 = 4500$ lbs. per sq. in. whence

$$t = \frac{10 \times 147}{4500} = 0.326 \text{ inches.}$$

But to insure safety in handling pipes and imperviousness to the water, a greater thickness is adopted in practice than given by the above theory.

Thus Weisbach recommends (as proved experimentally also) for

Not Homogen.

Pipes of sheet iron	$t = [0.00172\, rA + 0.12]$	inches
" " cast "	$t = 0.00476\, rA + 0.34$	" "
" " copper	$t = 0.00296\, rA + 0.16$	" "
" " lead	$t = 0.01014\, rA + 0.21$	" "
" " zinc	$t = 0.00484\, rA + 0.16$	" "

in which t = thicknes in inches, r = radius in inches, and A = excess of internal over external fluid pressure (i.e. $p - p_a$) expressed in *atmospheres*.

Thus, for the example just given, we would have (cast iron)

$$t = .004 \times 10 \times 10 + 0.34 = 0.816 \text{ inches}$$

If the pipe is subject to "water-ram" (§ 406) the strength should be much greater.

For a *THIN HOLLOW SPHERE* eq. (1) holds good.

For <u>thick</u> *hollow cylinders* see Rankine's Applied Mechanics p. 290; and Cotterill's Applied Mechanics p. 403.

409. COLLAPSING OF TUBES under FLUID PRESSURE

(Cylindrical boiler flues, for example.) If the external exceeds the internal fluid pressure, and the thickness of metal is small compared with the diameter, the slightest deformation of the tube or pipe gives the external greater capability to produce a further change of form and hence possibly a final collapse; just as with long columns (§ 303) a slight bending to the terminal forces. Hence the theory of § 408 is inapplicable. According to

Sir Wm. Fairbairn's experiments (1858) a thin wrought iron cylindrical (circular) tube will not collapse until the excess of external over internal pressure is

$$p\text{(lbs. per sq.in)} = 9672000\,\frac{t^2}{ld}\;\ldots\;(1)\ldots\text{(not. homog.)}$$

(t, l, and d all in same linear unit), in which t = thickness of the wall of the tube, d its diameter, and l its length; the ends being understood to be so supported as to preclude a local collapse.

Example. With $l = 10^{ft} = 120^{inches}$, $d = 4^{in}$ and $t = 1/10$ inch we have

$$p = 9672000\left[\frac{1}{100} \div (120 \times 4)\right] = 201.5 \ \text{lbs. per sq.in.}$$

For safety, $1/5$ of this, viz. 40 lbs. per sq. inch, should not be exceeded; e.g. with 14.7 lbs internal, and 54.7 lbs. external.

Chap. II. Hydrostatics, continued;

Pressure of Liquids in Tanks and Reservoirs.

410. BODY OF LIQUID IN MOTION, BUT IN RELATIVE EQUILIBRIUM. By relative equilibrium it is meant that the particles are not changing their relative positions, i.e., are not moving among each other. On account of this relative equilibrium the following problems are placed in the present chapter, instead of under the head of *Hydrodynamics*, where they strictly belong. As relative equilibrium is an essential property of rigid bodies, we may apply the equations of motion of rigid bodies to bodies of liquid in relative equilibrium.

Case I. All the particles moving in parallel right lines with equal velocities at any given instant; (i.e. a motion of TRANSLATION). If the common velocity is constant we have a *uniform translation*, and all the forces acting on any one particle are balanced, as if it were not moving at all (according to Newton's Laws, § 54); hence the relations of in-

ternal pressure, free surface, etc., are the same as if the liquid were at rest. Thus, Fig 469, if the liquid in the moving tank

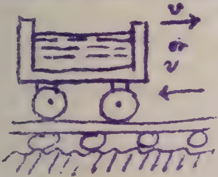


Fig 469

is at rest relatively to the **tank at** a given instant, with its free surface **horizontal**, and the motion of the tank be **one of translation** with a uniform velocity, the liquid will remain in this condition of relative rest, as the motion proceeds.

But if the velocity of the tank is *accelerated* with a constant acceleration $= \bar{p}$, (this symbol must not be confused with p for pressure), the free surface will begin to oscillate, and finally come to relative equilibrium at some angle α with the horizontal, which is thus found, when the motion is horizontal. Fig. 470, in which the position of α is equally applicable whether the motion is unif. accel. from left to right or uniformly retarded from right to left. Let O be the lowest point

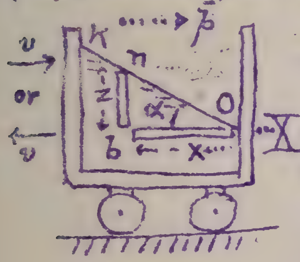


Fig. 470

of the free surface, and Ob a small prism of the liquid with its axis horizontal, and of length $= x$; nb is a vertical prism of length $= z$, and extending from the extremity of Ob to the free surface. The pressure at both O and n is p_a = atmos. press. Let cross section of both prisms be $= dF$

Now since Ob is being accelerated in direction X (horizont.) the difference between the forces on its two ends, i.e. its ΣX must = its mass X accel. (§ 109)

$$\therefore\ p_b\,dF - p_a\,dF = \left[x\,dF\gamma \div g\right]\bar{p} \quad \ldots\ldots (1)$$

(γ = heaviness of liquid; p_b = press. at b); and since the vertical prism nb has no vertical acceleration, the Σ vert. compons. for it must = 0

$$\therefore\ p_b\,dF - p_a\,dF - z\,dF\gamma = 0 \quad \ldots\ldots (2)$$

From (1) and (2) $\frac{x\gamma}{g}\bar{p} = z\gamma \therefore \frac{x}{z} = \frac{\bar{p}}{g}$ .....(3)

Hence **OnK** is a right line and $\therefore \tan\alpha = \frac{z}{x} = \frac{g}{\bar{p}}$ ..(4)

If the translation were vertical, and the acceleration upward [i.e., the vessel has a uniformly accelerated upward motion or a uniformly retarded downward motion] the free surface would be horizontal, but the pressure at a depth $= h$ below the surface instead of $p = p_a + h\gamma$ would be as follows: Considering free a small vertical prism of height $= h$ with upper base in the free surface, and putting Σ(vert. compons) = mass × acceleration, we have

$$dFp - dFp_a - hdF\gamma = \frac{hdF\gamma}{g}\bar{p} \therefore p = p_a + h\gamma\left[\frac{g+\bar{p}}{g}\right] \quad ..(5)$$

If the acceleration is downward (not the velocity necessarily) make $\bar{p}$ negative in (5). If the vessel falls freely, $\bar{p} = -g$ and $\therefore p = p_a$ in all parts of the liquid

Query. Suppose $\bar{p}$ downward and $> g$

Case II. Uniform Rotation about a vertical axis. If the narrow vessel in Fig. 471, open at top and containing a liquid, be kept rotating at a uniform angular velocity ω (see §110) about a vertical axis Z, the liquid after some oscillations will come to relative equilibrium (rotating about Z, as if rigid). Required the form of the free surface (evidently a surface of revolution) at each point of which $p = p_a$.

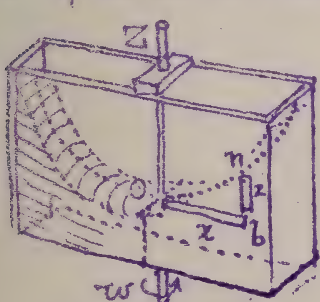


Fig. 471

Let O be the intersection of the axis Z with the surface and n any point in the surface; b being a point vertically under n and in same horizontal plane as O. Every point of the small right prism n.b (of altitude = z and section dF)

is describing a horizontal circle about Z and has ∴ no vertical acceleration. Hence for this prism, free, we have $\Sigma Z = 0$, i.e.

$$dF p_b - dF p_a - z\,dF\gamma = 0 \quad \ldots\ldots (1)$$

Now the horizontal right prism Ob (call the direction O..b, X) is rotating uniformly about a vertical axis thro' one extremity, as if it were a rigid body. Hence the forces acting on it must be equivalent to a single horizontal force $\omega^2 M \bar{p}$ (§ 122 a) coinciding in direction with X. [M = mass of prism = its weight ÷ g, and $\bar{p}$ = distance of its centre of gravity from O; here $\bar{p} = \frac{1}{2}x = \frac{1}{2}$ length of prism]. Hence the ΣX of the forces acting on the prism Ob must $= -\omega^2 \dfrac{x\,dF\gamma}{g}\dfrac{1}{2}x$

But the forces acting on the two ends of this prism are their own X compons., while the lateral pressures and the weights of its particles have no X compons.

$$\therefore\ dF p_a - dF p_b = \left(-\omega^2 x^2 dF\gamma\right) \div 2g \quad \ldots\ldots (2)$$

From (1) and (2) we have ..... $z = \dfrac{(\omega x)^2}{2g}$ ...(3)

Hence any vertical section of the free surface thro' the axis of rotation Z, is a parabola, with its axis vertical and vertex at O; i.e., the free surface is a paraboloid of revolution, with Z as its axis. Since ωx is the linear velocity v of the point b in its circular path, z = "height due to velocity" v.

Example. If the vessel in Fig. 471 makes 100 revol. per minute, required the ordinate z at a horizontal distance of x = 4 inches from the axis. (Ft. lb. sec. system) The angular velocity $\omega = [2\pi\, 100 \div 60]$ *radians per sec.* [N.B. A *radian* = the angular space of which 3.1415926... make a half-revol., or angle of 180°] With x = ⅓ ft. and g = 32.2

$$z = \frac{\omega^2 x^2}{2g} = \left(\frac{10\pi}{3}\right)^2 \left(\frac{1}{3}\right)^2 \frac{1}{64.4} = 0.188 \text{ ft.} = 2\tfrac{1}{4} \text{ inches}$$

and the pressure at b (Fig. 471) is (now use inch-lb-sec.)

$$p_b = p_a + z\gamma = 14.7 + 2\tfrac{1}{4} \times \frac{62.5}{1728} = 14.781 \text{ lbs. per sq. in}$$

Remark. If the vessel is quite full and closed on top, except at O′ where it communicates by a stationary pipe with a reservoir, Fig. 472, the free surface can not be formed but the pressure at any point in the water is just the same when uniform rotation goes on as if a free surface were formed with vertex at O i.e. $p_b = p_a + (h_0 + z)\gamma$ .....(4)

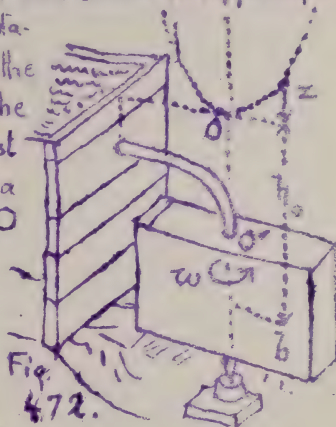


Fig. 472.

See figure for h_0 and z. [In subsequent §§ of this chapter, the liquid will be at rest]

411. PRESSURE ON BOTTOM.

If the bottom of the is plane and horizontal, the intensity of pressure upon it is the same at all points, being $p = p_a + h\gamma$, Figs. 473 and 474, and the pressures on the elements of the surface form a set of parallel (vertical) forces. This is true even if the side of the vessel overhangs, Fig. 474, the resultant fluid pressure on the bottom in both cases, being

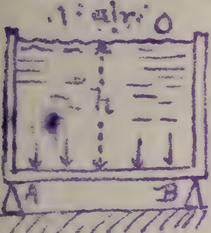


Fig. 473

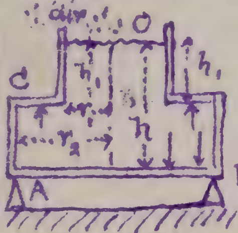


Fig. 474.

$$P = Fp - Fp_a = Fh\gamma \text{ ....(1)}$$

Atmos. press. is supposed to act under the bottom) It is further evident that if the bottom is a rigid homogeneous plate and has no support at its edge, it may be supported at a single point m Fig. 475, which in this case (horizontal plate) is its centre of gravity. This point is called the CENTER OF PRESSURE, or a point of application of the resultant of all the fluid pressures acting on the plate. The present case is such that these pressures reduce to a sin-

Fig. 475

the resultant, but this is not always practicable.

Example. In Fig. 474, (cylindrical vessel containing water), given $h = 20$ ft., $h_1 = 15$ ft., $r_1 = 2$ ft., $r_2 = 4$ ft., required the pressure on the **bottom**, the vertical tension in the cylindrical wall CA, and the **hoop** tension (§ 408) at C. (ft. lb. sec.)

Press. on bottom $= Fh\gamma = \pi r_2^2 h\gamma = \pi\, 16 \times 20 \times 62.5 = 62557.$ lbs.; while the upward pull on CA $= (\pi r_2^2 - \pi r_1^2) h_1 \gamma = \pi (16-4) 15 \times 62.5 = 32500$ lbs.

If the vertical wall is $t = \frac{1}{10}$ inch thick at C, this tension will be borne by a ring-shaped cross-section of area $= 2\pi r_2 t$ (nearly) $= 2\pi\, 48 \times \frac{1}{16} = 30.17$ sq. inches, giving $(32500 \div 30.17) =$ about 1000 lbs. per sq. inch tensile stress (vertical). The *hoop tension* at C is horizontal and is $p'' = r_2 (p - p_a) \div t$, (see § 408) where $p = p_a + h_1\gamma$, (in. lb. sec.)

$$\therefore p'' = \frac{48 \times 15 \times 12 \times (62.5 \div 1728)}{1/10} = 3125 \text{ lbs. per sq. in.}$$

412. CENTRE OF PRESSURE. In subsequent work in this chapter, since the atmosphere has access both to the free surface of liquid and to the outside of the vessel walls, and p_a would cancel out in finding the resultant fluid pressure on any elementary area dF of these walls, we shall write

The resultant fluid pressure on any dF of the vessel wall is normal to its surface and is $dP = p\,dF = z\gamma\,dF$, in which z is the vertical distance of the element below the free surface of the liquid (i.e. z = the "*head of water*") If the surface pressed on is plane, these elementary pressures form a system of parallel forces and may be replaced by a single resultant (if the plate is rigid) which will equal their sum, and whose point of application, called the CENTRE OF PRESSURE, may be located by the equations of § 22, put into calculus form.

If the surface is curved the elementary pressures form a system of forces in space, and hence (§ 38) cannot in general be reduced to a single resultant, but to two, the point of application of one of which is arbitrary (viz. the arbitrary origin, § 38)

Of course, the object of replacing a set of fluid pressures by a single resultant is for convenience in examining the equilibrium or stability of a rigid body the forces acting on which include these fluid pressures. As to their effect in distorting the rigid body, the fluid pressures must be considered in their true positions (See example in § 264)

413. RESULTANT LIQUID PRESSURE *on a plane surface forming part of a vessel wall. Co-ordinates of the* CENTER OF PRESSURE. Fig 476. Let AB be a portion (of any shape) of a plane surface at any angle, sustaining liquid pressure. Prolong the plane of AB till it intersects the free surface of the liquid. Take this intersection as an axis Y, O being any point on Y. The axis X, ⟂ to Y, lies in the given plane. Let α = angle between the plane and the free surface. Then x and y are the co-ordinates of any elementary area dF of the surface, referred to X and Y. z = the "head of water", below the free surface, of any dF. The pressures are ∥.

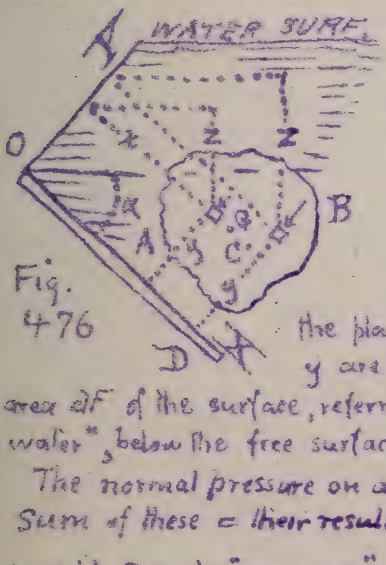


Fig. 476

The normal pressure on any dF, $= z\gamma\, dF$ ∴ the *Sum of these = their resultant* $= P = \gamma \int z\, dF = F\bar{z}\gamma$ ....(1)

in which $\bar{z}$ = the "*mean z*", i.e. the z of the centre of gravity G of the plane figure AB, and F = total area of AB. ($F\bar{z} = \int z\, dF$ from eq.(4) § 23). γ = heaviness of liquid.

That is, the total liquid pressure on a plane figure is equal to the weight of an imaginary prism of the liquid having a base = area of the given figure and an altitude

= depth (vertical) of the centre of gravity of the figure below the surface of the liquid. For example if the figure is a rectangle with one base (length = b) in the surface, and lying in a vertical plane, $P = bh \cdot \frac{1}{2} h \gamma = \frac{1}{2} bh^2 \gamma$; and if the altitude be increased, P varies as its *square*.

From (1) it is evident that the total pressure does not depend on the quantity of water in the reservoir.

Now let x_c and y_c denote the co-ordinates, in plane YOX, of the *centre of pressure*, C, or *point of application* of the resultant pressure P; then taking moments about OY (§ 22) we have

$$P x_c = \int_A^B (z\gamma dF)x \quad \text{and} \quad P y_c = \int_A^B (z\gamma dF)y \quad \ldots\ldots (2)$$

But $P = F\bar{z}\gamma = F\bar{x}(\sin\alpha)\gamma$, and the z of any $dF = x \sin\alpha$. Hence eqs. (2) become (after cancelling the constant, $\gamma \sin\alpha$)

$$x_c = \frac{\int x^2 dF}{F\bar{x}} = \frac{I_Y}{F\bar{x}} \quad \text{and} \quad y_c = \frac{\int xy\,dF}{F\bar{x}} \quad \ldots\ldots (3)$$

in which I_Y = the "mom. of inertia" of the plane figure referred to Y (see § 85) [N.B. The centre of pressure as thus found is identical with the centre of oscillation (§ 117) and the centre of percussion [§ 113] of a thin homogeneous plate, referred to axes X and Y, Y being the axis of suspension.]

Evidently, if the plane figure is vertical $\alpha = 90°$, $x = z$, and $\bar{x} = \bar{z}$. Also position of C of Pr. in the figure is independ. of α.

NOTE. Since the pressures on the equal dF's lying in any horizontal strip of the plane figure form a set of equal ∥ forces equally spaced along the strip, and are ∴ equivalent to their sum applied in the *middle* of the strip, it follows that for rectangles and triangles with horizontal bases, the centre of pressure must lie on the straight line on which the middles of all horizontal strips are situated.

414. CENTER OF PRESSURE OF RECTANGLES AND TRIANGLES WITH BASES HORIZONTAL. Since all

the dF's of one horizontal strip have the same x we may take the area of the strip for dF in the summation $\int x^2 dF$. $\therefore$ for the rectangle AB, Fig. 477., we have from eq. (3) § 413 with $dF = b\,dx$

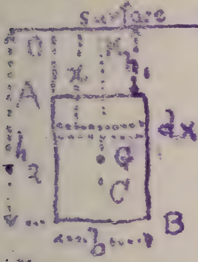


Fig. 477

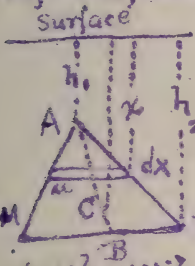
$$x_c = \overline{KC} = \frac{b\int_{h_1}^{h_2} x^2\,dx}{b(h_2-h_1)\frac{h_1+h_2}{2}} = \frac{2}{3}\,\frac{h_2^3-h_1^3}{h_2^2-h_1^2} \quad \ldots(1)$$

while (see NOTE § 413) $y_c = \frac{1}{2} b$

When the upper base lies in the surface, $h_1 = 0$, and $x_c = \frac{2}{3} h_2 = \frac{2}{3}$ of the altitude.

For a TRIANGLE with its base horizontal and vertex up, Fig. 478, the length u of a horizontal strip is variable and from similar triangles $u = \frac{b}{h_2-h_1}(x-h_1)$

$dF = u\,dx$



$$\therefore x_c = \frac{\int_{h_1}^{h_2} x^2\,dF}{F\bar{x}} = \frac{\frac{b}{h_2-h_1}\int_{h_1}^{h_2} x^2(x-h_1)\,dx}{\frac{1}{2}b(h_2-h_1)\left[h_1+\frac{2}{3}(h_2-h_1)\right]}$$

Fig. 478 But $\int x^2(x-h_1)\,dx = \left[\frac{x^4}{4} - h_1\frac{x^3}{3}\right]_{h_1}^{h_2}$

$$= \frac{1}{12}\left(3h_2^4 + h_1^4 - 4h_1h_2^3\right) = \frac{1}{12}(h_2-h_1)^2\left(3h_2^2+2h_1h_2+h_1^2\right)$$

$$\therefore x_c = \frac{1}{2}\cdot\frac{3h_2^2+2h_1h_2+h_1^2}{2h_2+h_1} \quad \ldots(2)$$

Also, since the C. of P. must lie on the line AB from the vertex to the middle of base (see NOTE § 413) we easily determine its position.

Similarly for a triangle with base horizontal and vertex down, Fig. 479, we find that

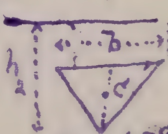
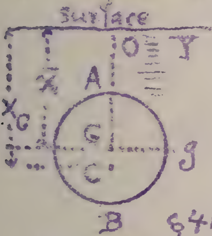


Fig. 479

$$x_c = \frac{1}{2}\cdot\frac{3h_1^2 + 2h_1h_2 + h_2^2}{2h_1 + h_2} \quad \ldots\ldots (3)$$

[If $h_1 = 0$ $x_c = \frac{1}{2}h_2$]

415. CENTER OF PRESSURE OF CIRCLE. Fig. 480. r = radius. It will lie on the vertical diameter AB



From eq. (3) § 413 we have

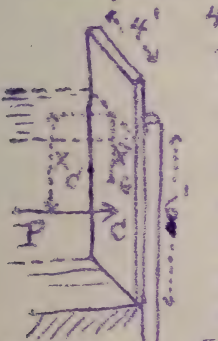
$$x_c = \frac{I_Y}{F\bar{x}} ; = \frac{I_g + F\bar{x}^2}{F\bar{x}} = \frac{\frac{1}{4}\pi r^4 + \pi r^2 \bar{x}^2}{\pi r^2 \bar{x}}$$

{See eq(4) § 88, and § 91} $\therefore x_c = \bar{x} + \frac{1}{4}\cdot\frac{r^2}{\bar{x}} \ldots (4)$

Fig. 480

§ 416 Examples. It will be noticed that although the total pressure on the plane figure depends for its value upon the head, $\bar{z}$, of the centre of gravity, its point of application is always lower than the centr. of gravity.

Example 1. If $6^{ft.}$ of a vertical sluice gate, 4^{ft} wide, Fig. 481 is below the water-surface, the total water pressure against it is (ft. lb. sec., eq.(1) § 413)



$$P = F\bar{z}\gamma = 6 \times 3 \times 62.5 = 1125 \text{ lbs.}$$

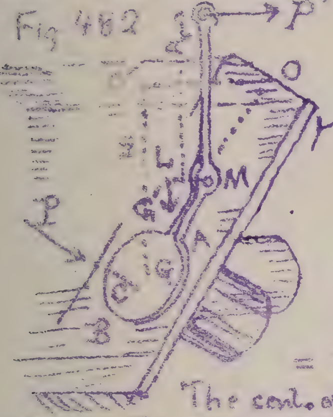
and (so far as the pressures on the vertical posts on which the gate slides are concerned) is equivalent to a single horizontal force of that value applied at a distance $x_c = \frac{2}{3}$ of $6 = 4$ ft below the surface (§ 414).

Fig. 481

Example 2. To (begin to) lift the gate in Fig. 481, the gate itself weighing 200 lbs., and the co-efficient of friction between the gate and posts being $f = 0.40$ (abstract numb.) (see § 156) we must employ an upward vertical force at least (ft. lb. sec.) $= P' = 200 + 0.40 \times 1125. = 4700$ lbs.

Example 3. Required the horizontal force P', Fig. 482, to be applied at N with a leverage of $a' = 30$ in. about

Fig. 482



the fulcrum M, necessary to (begin to) lift the circular disc AB of radius $r = 10$ in., covering an opening of equal size. NMAB is a single rigid lever, weighing $G' = 210$ lbs. The centre of gravity, G of disc being a vertical distance $\bar{z} = O'G = 40$ inches from the surface is 50 inches (viz. = sum of $OM = k = 20''$ and $MG = 30''$) from axis OY; i.e. $\bar{x} = 50$ in.

The cent. of grav. of whole lever is a horizontal distance $b' = LM = 12$ inches from M. For impending lifting we must have, for equil. of lever,

$$P'a' = G'b' + P(x_c - k) \quad \ldots\ldots \quad (1)$$

where P = total water press. on circular disk, and $x_c = OC$.

[Inch-lb. sec.] From eq (1) § 413,

$$P = F\bar{z}\gamma = \pi r^2 \bar{z}\gamma = \pi\, 100 \times 40 \times \frac{62.5}{1728} = 454.6 \text{ lbs.}$$

From § 415 $x_c = \overline{OC} = \bar{x} + \frac{1}{4}\frac{r^2}{\bar{x}} = 50 + \frac{1}{4}\cdot\frac{100}{50} = 50.5$ in.

$$\therefore P' = \frac{1}{a'}\left[G'b' + P(x_c - k)\right] = \frac{1}{30}\left[210 \times 12 + 454.6 \times 30.5\right] = 546 \text{ lbs.}$$

417. EXAMPLE OF FLOOD-GATE. Fig. 483. Supposing the rigid double gate AD, 8 ft. in total width to have four hinges, two at e, and two at f, 1 ft from top and bottom of water channel, required the pressures upon them, taking dimensions from the figure (ft. lb. sec.)

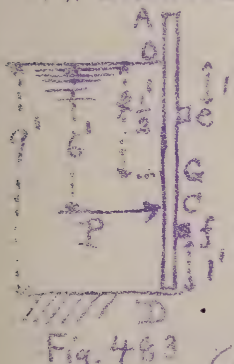


Fig. 483

Wat. press. $= P = F\bar{z}\gamma = 72 \times 4\frac{1}{2} \times 62.5 = 20250.$ pounds, and its pt. of applic. (cent. of press) is a distance $x_c = \frac{2}{3}$ of $9' = 6'$ from O (§ 414) Considering the whole gate free and taking moments about e, we shall have

(press. at f) $\times 7' = 20250 \times 5'$ $\therefore$ press. at $f = 14464$. lbs (half on each hinge at f) and $\therefore$ press. at e = P − press. at f = 5875. lbs. {half of which comes on each hinge

418. STABILITY OF A VERTICAL RECTANGULAR WALL against WATER PRESSURE ON ONE SIDE. Fig. 484.

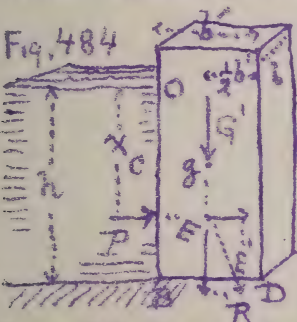


Fig. 484

Suppose the wall to be a single rigid block, its weight $G' = b'h'l\gamma'$; (γ' being its heaviness, § 7, and l its length). Given the water depth $= h$, required the proper width $b' = ?$ for stability. We must have *First*, the resultant of G' and the water press. P must fall within the base BD, (or, which amounts to the same thing, the moment of G' about D, the outer *toe* of the wall, must be numerically greater than that of P; and *Secondly*, P must be less than the sliding friction fG' (see § 156) on the base BD. *Thirdly* the max. press. per unit of area on the base must not exceed a safe value (Compare §)

Now $P = F\bar{z}\gamma = hl\frac{h}{2}\gamma = \frac{1}{2}h^2l\gamma$, ($\gamma$ = for water) and $x_c = \frac{2}{3}h$. Hence for stability against tipping about D,

$P\frac{1}{3}h$ must be $< G'\frac{1}{2}b'$, i.e., $\frac{1}{6}h^3l\gamma < \frac{1}{2}b'^2h'l\gamma'$ ....(1)

while, as to sliding on the base,

P must be $< fG'$, i.e., $\frac{1}{2}h^2l\gamma < fb'h'l\gamma'$ ....(2)

419. DITTO; MORE DETAILED SOLUTION. If

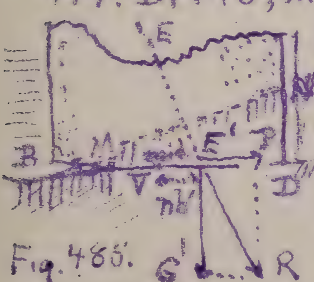


Fig. 485.

(1) were an exact equality instead of an inequality the resultant R of P and G' would pass through the corner D, tipping would be impending and the pressure per unit area at D would be infinite. To avoid this we wish the wall to be wide enough that the re-

sultant R, Fig. 465, may cut BD in such a point E', that the pressure per unit area, p_m, at D shall have a definite safe value (being the max. press.) This may be done by the principles §§ 346 and 362.

First, assume that R cuts BD outside of the middle third, i.e. that $VE = nb'$ is $> \frac{1}{6} b'$ (or $n > 1/6$) Then the pressures on small equal elements of the base BD (see § 346) vary as the ordinates of a triangle MND, and ED will $= \frac{1}{3}$ MD; i.e. $\overline{MD} = 3\left(\frac{1}{2} - n\right) b'$ The mean press., per unit area, on MD $= G' \div l.\overline{MD}$ and hence the max. press., viz. at D, being double the mean is $p_m = 2\, G' \div \left[3\, b' l \left(\frac{1}{2} - n\right)\right]$ and if p_m is to equal C' (see §§ 201 and 203) a safe value for the crushing resistance, per unit area, of the material, we shall have

$$b' l\left(\tfrac{1}{2} - n\right) C' = \tfrac{2}{3} G' = \tfrac{2}{3} b' h' l \gamma', \therefore\ n = \frac{1}{2} - \frac{2}{3}\frac{h'\gamma'}{C'} \quad \ldots(1)$$

Knowing n, to find b', we put the Σ(moms.) of the G' and P at E, about E' = zero, i.e. $G' n b' - P \frac{1}{3} h = 0$

$$\text{or, } n b'^2 h' l \gamma' = \tfrac{1}{3} h \cdot \tfrac{1}{2} h^2 l \gamma \therefore\ b' = h \cdot \sqrt{\frac{h\gamma}{6 n h' \gamma'}} \quad \ldots(2)$$

Having obtained b', we must also ascertain if P is $< fG'$ the friction, i.e. if P is $< f\, b' h' l \gamma'$. If not b' must be still further increased.

If n from (1) should prove to be $< \frac{1}{6}$, our first assumption is wrong and we $\therefore$ assume $n < 1/6$ and proceed thus:

Secondly, n being $< \frac{1}{6}$ (see §§ 346 and 362) we have a trapezoid of pressures, instead of a triangle, on BD Let the press. per unit area at D be p_m (the max. on base) The mean " " " " on BD being now $= \left\{ \begin{array}{l} G' \div b' l \\ = h' \gamma' \end{array} \right.$ we have (§ 362) $p_m = (6n + 1) h' \gamma'$.

For safety we put $(6n+1)h'\gamma' = C'$ } whence $n = \frac{1}{6}\left[\frac{C'}{h'\gamma'} - 1\right]$ ....(1a)

and then use eq. (2) for b'

Example. In Fig. 484, let $h' = 12$ ft., $h = 10$ ft, while the masonry weighs ($\gamma' =$) 150 lbs. per cub. ft. Supposing it desirable to bring no greater compressive stress than 100 lbs. per sq. inch (= 14400 lbs. per sq. ft.) on the cement of the joints, we put $C' = 14400$, and use the ft. lb. sec. system.

Trying eq. (1) } $n = \frac{1}{2} - \frac{2}{3}\cdot\frac{12\times 150}{14400} = 5/12$ which is $> \frac{1}{6}$

∴ from eq. (2) we have } $b' = 10\times\sqrt{\frac{62.5\times 10}{\frac{5}{2}\times 12\times 150}} = 3.7^{ft}$ { as a proper thickness

If we make $n = 1/6$, i.e. make R cut the base at the outer edge of middle third (§ 362) we would have

$b' = 10\times\sqrt{\frac{62.5\times 10}{6/6\times 12\times 150}} = 5.89$ ft { which for so small a wall is preferable to 3.7 ft of the other case which brings R a little too near the corner D.

As regards frictional stability, we find that with $f = 0.30$ a low value, and $b' = 3.7$ ft ~~we find that~~ (ft. lb. sec.)

$$\frac{P}{fG} = \frac{\frac{1}{2}h^2\gamma}{f\,b'h'\gamma'} = \frac{100\times 62.5}{2\times 0.3\times 3.7\times 12\times 150} = \frac{3}{2},$$

which is $>$ unity, showing the friction to be insufficient to prevent sliding (with $f = 0.30$); but with $b' = 5.89$ ft. we find this ratio to be 0.92, indicating frictional stability.

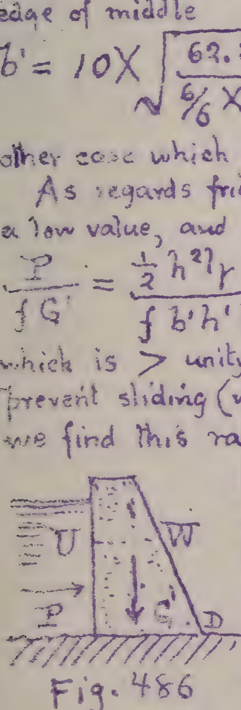


Fig. 486

Economy of material is favored by using a trapezoidal profile Fig. 486. With this form the stability may be investigated in a corresponding manner. The portion of wall above each horiz. bed UW, should be examined similarly. The weight G' acts thro' the centre of gravity of whole mass. [See § 26.]

420. HIGH MASONRY DAMS. Although the principle of the arch may be utilized for vertical stone dykes of small height (20 or 30 feet) for greater heights the formula for hoop tension, §408, (or rather, here, hoop compression, on the vertical radial joints of the horizontal arch rings, Fig. 487) gives such large values in the lower courses, and would thereby call for so great a radial thickness of joint in those courses, that straight dykes are usually built instead, even where firm rock abutments are available laterally.

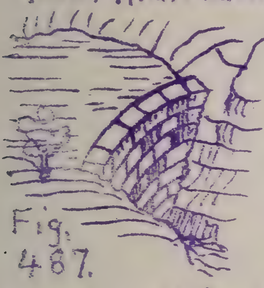


Fig. 487.

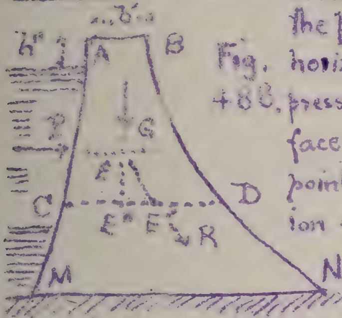
For example, at a depth of 100 feet, where the hydrostatic pressure is $h\gamma = 100 \times 62.5 = 6250.$ lbs. per sq. ft., if the are to have a (radial, horizontal) thickness $= 4$ ft with a horizon. radius of curvature $r = 100$ feet, we shall find a compression between their vert., radial faces of (ft.lb.sec.)

$$p'' = \frac{r(p - p_a)}{t} = \frac{100 \times 6250}{4} = 156250. \text{ lbs. per sq. ft.}$$

or 1085 lbs. per sq. inch, far too great for safety, even if there were no danger of collapse, the dyke being short.

Also, if the courses were 4 feet thick at all other depths, the pressure on the horizontal faces would be (supposing γ' for the stone $= 150$ lbs. per cub. foot) $100 \times 150 = 15000$ lbs. per sq. ft. $=$ 104 lbs. per sq. in., which might be more than desirable.

Fig. 488 shows the profile of a straight high masonry dam as designed at the present day, (from 60 to 160 feet high) Assuming a width $b' =$ from 6 to 16 feet at the top, and a sufficient h'' (see figure) to exceed the max. height of waves, the up stream outline ACM is made nearly vertical, and perhaps somewhat concave, while the down-stream profile BDN, by computation or graphical trial is so formed that when the reservoir is full the resultant R, of the weight G of

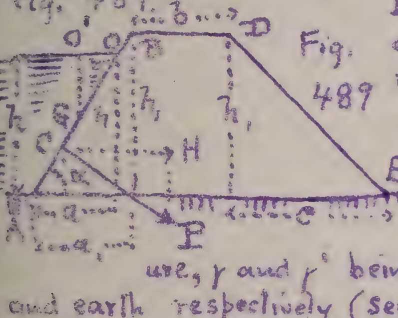


the portion ABCD of masonry above each horizontal bed, as CD, and the hydrostatic pressure P on the corresponding up-stream face AC, shall cut the bed CD in such a point E' as not to cause too great compression at the outer edge D (not over 85 lbs. per sq. inch according to M. Kranz in "*Reservoir Walls*") this being computed as in § 419. Nor, when the reservoir is empty and the water press. lacking, must the weight G resting on each bed, as CD, cut the bed in a point E'' so near the edge C as to produce excessive pressure there. The figure shows the general form of profile resulting from these conditions. The masonry should be of such a character, by irregular bonding in every direction, as to make the wall if possible a monolith.

Fig. 488.

421. EARTHWORK DAM OF TRAPEZOIDAL SECTION

Fig. 489.



It is required to find the conditions of stability of the straight earthwork dam ABDE, whose length = l, ⊥ to paper, as regards sliding horizontally on the plane AE; i.e. its frictional stability. With the dimensions of the figure, γ and γ' being the heavinesses of the water and earth respectively (see § 7), we have

$$\text{Weight of dam} = G' = \text{vol} \times \gamma' = l h_1 \left[b + \tfrac{1}{2}(a_1 + c) \right] \gamma' \quad \ldots (1)$$

$$\text{Resultant water press.} = P = F \bar{z} \gamma = \overline{OA} \times l \times \tfrac{1}{2} h \gamma \quad \ldots (2)$$

$$\text{Horiz. comp. of } P = H = P \sin\alpha = \left[\overline{OA} \sin\alpha \right] \tfrac{1}{2} h l \gamma = \tfrac{1}{2} h^2 l \gamma \quad \ldots (3)$$

From (3) it is evident that the horizon. component of P is just the same, viz.: $hl \cdot \tfrac{1}{2} h\gamma$, as if the water pressure would be on a vertical rectangle equal to the ver-

tical projection of OA and with its centre of gravity at the same depth $(\frac{1}{2}h)$ Compare § 400. Also

Vert. comp of $P = V = P\cos\alpha = \left[\overline{OA}\cos\alpha\right]\frac{1}{2}hl\gamma = \frac{1}{2}ahl\gamma$ ...(4)

and is the same as the water pressure on the horizontal projection of OA if placed at a depth $= O'G = \frac{1}{2}h$.

For stability against sliding the horiz. compon. of P must be less than the friction due to the total vertical pressure on the plane AE viz. $G_1 + V$; hence if f is the co-efficient of friction on AE we must have $H < f[G_1 + V]$ i.e., see above,

$$\frac{1}{2}h^2 l\gamma \text{ must be} < f\left[lh_1\left(b+\tfrac{1}{2}(a_1+c)\right)\gamma' + \tfrac{1}{2}ahl\gamma\right] \ldots (5)$$

However, if the water leaks under the dam on the surface AE, so as to exert an upward hydrostatic pressure $V' = [a_1+b+c]lh\gamma$ the friction will be only $= f[G_1 + V - V']$ and (5) will be replaced by

$$H < f[G_1 + V - V'] \ldots\ldots\ldots (6)$$

Experiment shows (Weisbach) that with $f = \cancel{0.30}\ 0.33$ computations made from (5) and (6) (treated as bare equalities) give satisfactory results.

Example. (ft. lb. sec.) With $f = \cancel{0.30}\ 0.33$, $h = 20$ ft, $h_1 = 22$ ft, $a = 24$ ft, $a_1 = 26.4$ ft, and $c = 30$ ft. we have making (6) an equality, with $\gamma' = 2\gamma$,

$$\frac{1}{2}h^2 l\gamma = f\left[\gamma' lh_1\left(b + \frac{a_1+c}{2}\right) + \frac{1}{2}ahl\gamma - (a_1+b+c)lh\gamma\right]$$

$$\frac{1}{2}400 = \frac{1}{3}\left[22(b+28.2)2 + \tfrac{1}{2}24\times 20 - (26.4+b+30)20\right]$$

whence, solving for b, the width of top, $b = 10.3$ feet.

422. LIQUID PRESSURE ON BOTH SIDES OF A GATE OR RIGID PLATE. The sluice-gate AB, for example, Fig. 490, receives a pressure, P_1, from the "head-water" M, and an opposing pressure P_2 from the "tail

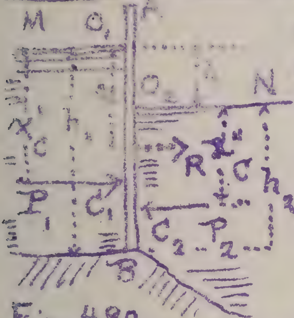


Fig. 490

water N. Since these two horiz. forces are not in the same line, though parallel, their resultant R, which $= P_1 - P_2$, acts horizontally in the same plane but at a distance below $O_1 = u$, which we may find by placing the moment of R about O_1, equal to the algebraic sum of those of P_1 and P_2 about O_1

$$\therefore \quad Ru = P_1 x'_c - P_2 (x''_c + h) \quad \dots \quad (1)$$

whence $$u = \left[P_1 x'_c - P_2 (x''_c + h)\right] \div \left[P_1 - P_2\right] \quad \dots (2)$$

C_1 and C_2 are the respective centres of pressure of the surfaces O_1B and O_2B, and u = distance of R from O_1, while h = difference of level between head and tail waters. If the surfaces O_1B and O_2B are both rectangular,

$$x'_c = \frac{2}{3}h_1 \quad \text{and} \quad x''_c = \frac{2}{3}h_2$$

Example. Let the dimensions be as in Fig. 491, both surfaces under pressure being rectangular and 8 ft. wide. Then (ft. lb. sec.)

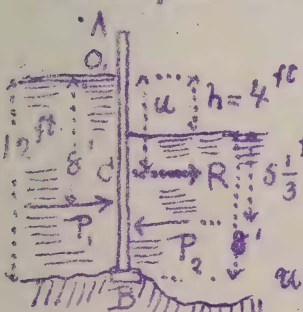


Fig. 491

$$R = P_1 - P_2 \quad \text{or} \quad , \S\, 416,$$

$$R = [12 \times 8 \times 6 - 8 \times 8 \times 4]\, 62.5 = 20000.$$

lbs. = 10 tons; while from eq. (2)

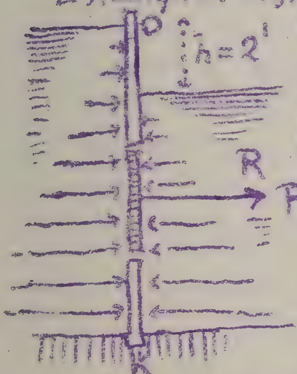
$$u = \frac{[12 \times 8 \times 6 \times 8 - 8 \times 8 \times 4 (9\tfrac{1}{3})]\, 62.5}{20000}$$

That is, $u = 6.93$ feet which locates C. Hence the pressure of the gate upon its hinges or other support is the same (aside from its own weight), provided it is rigid, as if the single horizontal force $R = 10$ tons acted at the point C 2.93 ft. below the level of the tail-water surface.

If the plate, or gate, is entirely below the tail water

surface the resultant pressure is applied in the centre of gravity of the plate. PROOF as follows: Conceive the surface to be divided into a great number of small *equal* areas, each $= dF$; then, the head of water of any dF being $= x_1$ on the head-water side, and $= x_2$ on the tail-water side, the resultant pressure on the dF is $\gamma dF(x_1 - x_2) = \gamma h dF$, in which h is the difference of level between head and tail water. That is, the ~~resultant pressures~~ on the equal dF's are *equal*, and hence form a system of equal parallel forces distributed over the plate in the same manner as the weights of the corresponding portions of the plate; ∴ their single resultant acts through the centre of gravity of the plate; Q.E.D. This single resultant $= \int \gamma h dF = \gamma h \int dF = Fh\gamma$.

Example. Fig. 492. The resultant pressure on a circular disk of radius $= 8$ inches, in the vertical partition CK, which has its centre of gravity 3 ft. below the tail water surface, with $h = 2$ ft., is (ft. lb. sec.)



$$R = Fh\gamma = \pi r^2 h\gamma = \pi\, 8^2 \times 24 \times \frac{62.5}{1728}$$

$= 174.6$ lbs., and is applied through the centre of gravity of the circle. *Evidently R is the same for any depth below the tail water surface, so long as $h = 2$ ft.*

Fig. 492. [Let the student find a graphic proof of this §]

423. LIQUID PRESSURE ON CURVED SURFACES.

If the rigid surface is curved, the pressures on the individual dF's, or elements of area, do not form a system of parallel forces and the single resultant (if one is practicable) is not equal to their sum. In general, the system is not equivalent to a single force, but can always be reduced to two forces (§ 38) the point of application of one of which is arbitrary (the arbitrary origin of § 38) and its amount $= \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$

A single *Example* will be given; that of a thin rigid shell having the shape of the curved surface of a right cone, Fig. 493, its altitude being h and radius of base $= r$. It has no bottom, is placed on a smooth horiz. table, vertex up, and is filled with water thro' a small hole in the apex O, which is left open (to admit atmospheric pressure). What load, besides its own weight G', must be placed on it to prevent the water from lifting it and escaping under the edge A ?

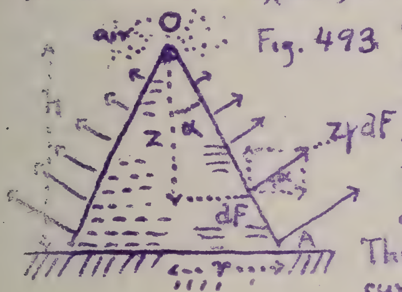


Fig. 493

The pressure on each dF of the inner curved surface is $z\gamma\, dF$ and is normal to the surface. Its vertical component is $z\gamma dF \sin\alpha$ and horizontal compon. $= z\gamma dF \cos.\alpha$. The dF's have all the same α but different z's (or heads of water). The lifting tendency of the water on the thin shell is due to the vertical components forming a system of ∥ forces, while the horizontal components, radiating symmetrically from the axis of the cone, neutralize each other. Hence the resultant lifting force is $V = \Sigma(\text{vert. comps.}) = \gamma \sin\alpha \int z\,dF = \gamma \sin\alpha\, F\bar{z} \ldots (1)$ where F = total area of curved surface and $\bar{z}$ the "head of water" of its centre of gravity. Eq. (1) may also be written thus:

$V = \gamma F_b \bar{z} \ldots\ldots (2)$ in which $F_b = F\sin\alpha$ = area of the circular base = area of the *projection* of the curved surface upon a plane ⊥ to the vertical, i.e. upon a horizontal plane. Hence we may write

$V = \frac{2}{3}\gamma\pi r^2 h \ldots (3)$ $\Big\{$ since $\bar{z} = \frac{2}{3}h$ being the z of the centre of gravity of the curved surface and not that of the base. γ = heaviness of water.

If G' = weight of the shell and is $< V$, an additional load $G_1 = G' - V$ would be needed to prevent the lifting.

If the shell has a bottom forming a base for the cone and

of weight $= G''$, we find that the vertical liquid forces acting on the whole rigid body, base and wall, are V upward, G' and G'' downward and the liquid pressure on the base, viz. $V' = \pi r^2 h\gamma$ (§ 411) also downward. Hence the resultant vertical force to be counteracted by the table is downward and $= G' + G'' + V' - V$, which $= G' + G'' + \frac{1}{3}\pi r^2 h\gamma$ ...(4) i.e., the total weight of the rigid vessel and the water in it, as we know of course in advance.

Chap. II

STATICS OF IMMERSION AND FLOTATION.

424. RIGID BODY IMMERSED IN A LIQUID. BUOYANT EFFORT. If any portion of a body of homogeneous liquid at rest be conceived to become rigid without alteration of shape or bulk, it would evidently still remain at rest; i.e., its weight, applied at its centre of gravity would be balanced by the pressures, on its bounding surfaces, of the contiguous portions of the liquid; hence

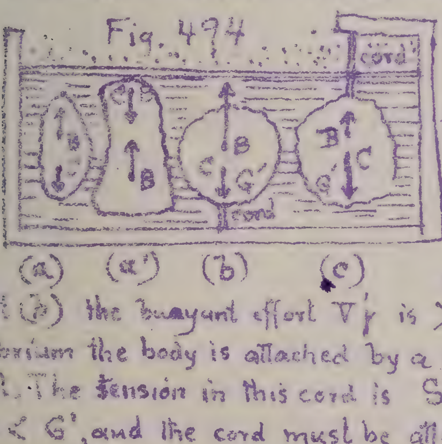
If a rigid body or solid is immersed in a liquid, both at rest, the resultant action upon it of the surrounding liquid (or fluid) is a vertical upward force called the BUOYANT EFFORT, *equal in amount to the weight of liquid displaced, and acting through the centre of gravity of the volume (considered as homogeneous) of displacement (now occupied by the solid).* This point is called the CENTRE OF BUOYANCY, and is sometimes spoken of as *the centre of gravity of displaced water.* If V' = the volume of displacement and γ = heaviness of the liquid, then $\left\{ \text{BU. EFFORT} = V'\gamma \right.$

If the weight G' of the solid is not equal to the buoyant effort, or if its centre of gravity does not lie in the same vertical as the centre of buoyancy, the

two forces form an unbalanced system and motion begins. But as a consequence of this very motion the action of the liquid is modified in a manner dependent on the shape and kind of motion of the body. Problems in this chapter are restricted to cases of rest, i.e. balanced forces. Suppose $G' = V'\gamma$, then:

If the centre of gravity lies in the same as the centre of buoyancy and *underneath* the latter, the equilibrium is stable, i.e. after a slight angular disturbance the body returns to its original position (after several oscillations); while if *above* the latter, the equilibrium is unstable. If they co-incide, as when the solid is homogeneous and of the same heaviness as the liquid, the equil. is indifferent, i.e., possible in any position of the body

424a. EXAMPLES OF IMMERSION. Fig. 494. At



(a) (a') (b) (c)

(a) is an example of stable equilibrium, the centr of buoyancy B being above the centre of grav. C, and the buoyant effort $V'\gamma = G'$ the weight of the solid; at (a'), conversely, we have unstable equilibrium, with $V'\gamma$ still $= G'$. At (b) the buoyant effort $V'\gamma$ is $> G'$, and to preserve equilibrium the body is attached by a cord to the bottom of the vessel. The tension in this cord is $S = V'\gamma - G'$. At (c) $V'\gamma$ is $< G'$, and the cord must be attached to a support above and its tension is

$$S = G' - V'\gamma \quad \text{.........} \quad (1)$$

If in eq. (1) [(c) in Fig.] we call S the *apparent weight* of the immersed body, and measure by a spring-, or beam-balance, we may say that

The apparent weight of a solid totally immersed in a liquid equals its real weight diminished by that of the amount of liquid displaced; in other words, *the loss of*

weight = the weight of displaced liquid.

Example 1. How great a mass (not hollow) of cast iron can be supported in water by a wro't iron cylinder weighing 140 lbs., if the latter contains a vacuous space and displaces 3 cub. feet of water, both bodies completely immersed? [ft. lb. sec.]

The buoyant effort on the cylinder is $V'\gamma = 3 \times 62.5 = 187.5$ lbs., leaving a residue 47.5 lbs. upward force to buoy the cast iron, whose volume V'' is unknown, while its heaviness (§ 7) is $\gamma'' = 450$ lbs. per cub. foot. The direct buoyant effort of the water on the cast iron is $V''\gamma = [V'' \times 62.5]$ lbs. and the problem requires that this force + 47.5 lbs. shall = $V''\gamma''$ the weight G'' of the cast iron $\therefore V'' \times 62.5 + 47.5 = V'' \times 450 \;\therefore V'' = 0.12$ cub. ft.

while $0.12 \times 450 = 54.$ lbs. of cast iron. *Ans.*

Example 2. Required the volume V', and heaviness γ', of a <u>homogeneous</u> solid which weighs 6 lbs. out of water and 4 lbs. when immersed (*apparent weight*). (ft. lb. sec.)

From eq. (2) $4 = 6 - V'62.5 \;\therefore V' = 0.032$ cub. feet

$\therefore \gamma' = G' \div V' = 6 \div 0.032 = 187.5$ lbs. per cub. ft.

and the ratio of γ' to γ is $187.5 : 62.5 = 3.0$ (abstr. numb.) i.e. the substance of this solid is 3 times as dense, or three times as heavy as water. [The buoyant effort of the air has been neglected in giving the true weight as 6 lbs.]

425. SPECIFIC GRAVITY. By *specific gravity* is meant the ratio of the heaviness of a given homogeneous substance to that of a standard homog. substance; in other words, the ratio of the weight of a certain volume of the substance to the weight of an *equal volume* of the standard substance. Distilled water at the temperature of max. density (4° Centigrade) under a pressure of 14.7 lbs. per sq. inch is sometimes taken as the standard substance, more frequently however at

62° Fahrenheit (16.6° Centig.). Water, then, being the standard substance, the numerical example last given illustrates a common method of determining experimentally the spec. grav. of a homogeneous solid substance, the value there obtained being 3. The symbol σ (small Greek sigma) will be used to denote specific gravity which is evidently an abstract number. The standard substance should always be mentioned, and its heaviness γ; then the heaviness of a substance whose spec. grav. is σ is $\gamma' = \sigma\gamma$ . . . . (1)

and the weight G' of any volume V' of the substance may be written
$$G' = V'\gamma' = V'\sigma\gamma \quad \ldots\ldots (2)$$

Evidently a knowledge of the value of γ' dispenses with the use of σ, though when the latter can be introduced into problems involving the buoyant effort of a liquid the criterion as whether a *homogeneous* solid will sink or rise, when immersed in the *standard* liquid, is more easily applied, thus: Being immersed the volume V' of the body = that, V, of displaced liquid. Hence

if G' is $> V'\gamma$, i.e., if $V'\gamma'$ is $> V'\gamma$, or $\sigma > 1$, it sinks,
while if G' is $< V'\gamma$ .. .... or $\sigma < 1$. it rises;
i.e. according as the weight G' is $>$ or $<$ than the buoy. effort.

Other methods of determining the spec. grav. of solids, liquids and gases are given in works on Physics.

426 EQUILIBRIUM OF FLOTATION. In case the weight G' of an immersed solid is $<$ the buoyant effort $V\gamma$ (where V is the volume of displacement and γ the heav. of liquid) the body rises to the surface and after a series of oscillations comes to rest in such a position, Fig. 495, that its centre of gravity C and the centre of buoyancy B (the new B, belonging to the new volume of displacement, which is limited above by the horiz. plane of the free surface of the liquid) are in the same vertical (called the axis of flotation) (or line of support) and

that the volume of displacement has diminished to such a new value V, that $V\gamma = G'$. ....(1)

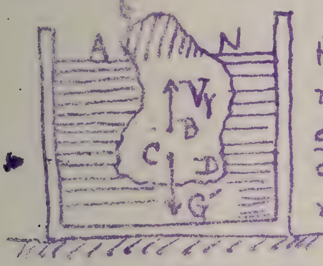


Fig. 495.

In the figure V = vol. AND, below the horiz. plane AN, and the slightest motion of the body will change the form of this vol., in general, (whereas with complete immersion the vol. of displacement remains constant). For stable equilibrium it not essential in every case that C (cent. of grav. of body) should be below B (the centre of buoyancy) as with complete immersion, since if the solid is turned B may change its position in the body, as the form of the vol. AND changes.

There is now no definite relation between the vol. of displacement V and that of the body, V', unless the latter is homogeneous, and then for G' we may write $V'\gamma'$, i.e.

$$V'\gamma' = V\gamma \quad \text{(for homogen. solid)} \ldots\ldots (2)$$

or, the volumes are inversely proportional to the heavinesses.

The buoyant effort of the air on the portion ANE may be neglected, as insignificant, in most practical cases.

If the solid is hollow, the position of its centre of gravity C may be easily varied (by shifting ballast e.g.) within certain limits, but that of the centre of buoyancy B depends only on the geometrical form of the vol. of displacement AND below the horizontal plane AN

Example. Ft. lb. sec. A solid weighing $G' = 400$ lbs. and having a volume $V' = 8$ cub. feet, without hollows or recesses, will float in water; since to obtain a buoyant effort of 400 lbs., we need a vol. of displacement, see eq. (1), of $V = \frac{G'}{\gamma} = 400 \div 62.5 =$ only 6.4 cub. ft. Hence the solid will float with $8 - 6.4$ or 1.6 cub. ft. projecting above the water level.

427. THE HYDROMETER is a floating instrument for determining the relative heavinesses of liquids. Fig. 496 shows the simplest form, consisting of a bulb and cylindrical stem of glass, so designed and weighted as to float upright in all liquids whose heavinesses it is to compare. Let F denote the uniform sectional area of the stem (a circle), and suppose that when floating in water (whose heav. $= \gamma$) the water surface marks a point A on the stem; and that when floating in another liquid, say petroleum, whose heav. $= \gamma_p$ we wish to determine, it floats at a greater depth, the liquid surface now marking A' on the stem, a height $= x$ above A.

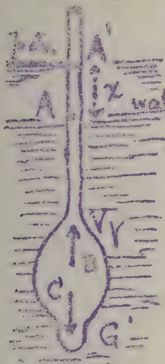


Fig. 496.

G' is the same in both experiments, but while the vol. of displacement in water is V, in petroleum it is $V + Fx$. ∴ eq. (1), § 425, in the water $G' = V\gamma$ . . . . (1)

and in the petroleum $G' = (V + Fx)\gamma_p$ ... (2)

from which, knowing G', F, x, and γ we find V and γ_p

i.e. $V = \dfrac{G'}{\gamma}$ and $\gamma_p = \dfrac{G'\gamma}{G' + Fx\gamma}$ ....(3)

[N.B: F is best determined by noting the additional distance l through which the instrument sinks in water under an additional load P, not immersed; for then $G' + P = (V + Fl)\gamma$ ∴ $F = \dfrac{P}{l\gamma}$]

Example. [Using the inch, ounce, & second, in which system $\gamma = 1000 \div 1728 = 0.578$ oz. (§ 394)] With $G' = 3$ ounces, and $F = 0.10$ sq. inches, x being observed, on the graduated stem, to be 5 inches we have for the petroleum

$$\gamma_p = \frac{3 \times 0.578}{3 + 0.10 \times 5 \times 0.578} = 0.525 \text{ oz. per cubic inch} = 56.7 \text{ lbs. per cub. feet.}$$

Temperature influences the heaviness of most liquids to some extent.

428. DEPTH OF FLOTATION. The weight and exterior shape of the floating body are known, and the centre of gravity so situated that the position of flotation is known the depth of the lowest point below the surface may be determined.

Case I. Right prism or cylinder with its axis vertical.

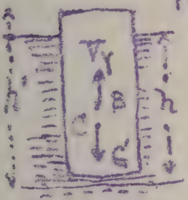


Fig. 497.

Fig. 497. (For stability in this position see § 430) Let G' = weight of cylinder, F the area of its cross section (full circle), h' its altitude, and h the unknown depth of flotation (or draught); then from eq. (1) § 426

$$G' = Fh\gamma \quad \therefore \quad h = \frac{G'}{F\gamma} \quad \ldots\ldots (1)$$

in which γ = heav. of the liquid. If the prism (or cylin.) is homogeneous (and then C, at the middle of h', is higher than B) and γ' its heaviness we have

$$h = \frac{Fh'\gamma'}{F\gamma} = \frac{\gamma'}{\gamma}h' = \sigma h' \ldots (2)$$

see § 425

in which σ = specif. grav. of solid referred to the liquid as standard.

Case II. Pyramid or cone with axis vertical and vertex down. Fig. 498. Let V' = vol. of whole pyramid (or cone, and V = vol. of displacement. From similar pyramids $\frac{V}{V'} = \frac{h^3}{h'^3}$ $\therefore h = \sqrt[3]{\frac{V}{V'}}\cdot h'$

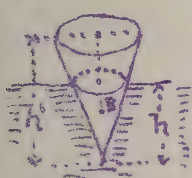


Fig. 498

But $G' = V\gamma$ or, $V = \frac{G'}{\gamma}$ $\Big\}$ $\therefore h = h'\sqrt[3]{\frac{G'}{V'\gamma}} \ldots\ldots (3)$

Case III, Ditto but vertex up. Fig 499, Let the notation be as before, for V and V'. The part out of water is a pyramid, vol. $= V'' = V' - V$, and is similar to the whole pyramid, $\therefore$ $V' - V : V' :: h''^3 : h'^3$; also $V = \frac{G'}{\gamma}$

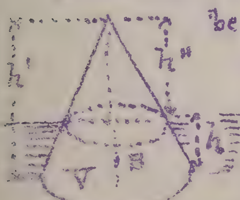
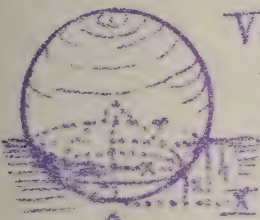


Fig. 499.

$$\therefore h'' = h'\sqrt[3]{\frac{V' - V}{V'}} = h'\sqrt[3]{\frac{V'\gamma - G'}{V'\gamma}} \text{ and}$$

$\therefore$ finally $h = h'\left[1 - \sqrt[3]{1 - [G' \div V\gamma]}\right]$ .... (4.)

Case III. *Sphere.* Fig. 500. The volume immersed is



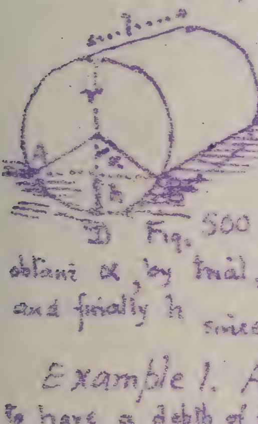
$$V = \int_{z=0}^{z=h} (\pi x^2)\,dz = \pi\int_0^h (2rz - z^2)\,dz = \pi h^2\left[r - \frac{h}{3}\right]$$

and $\therefore$ since $V\gamma = G'$ = weight of sphere,

$$\pi r h^2 - \frac{\pi h^3}{3} = \frac{G'}{\gamma} \quad \ldots\ldots\ldots (5.)$$

Fig. 500. From which cubic equation h may be obtained by successive trials and approximations.

Case IV. *Right cylinder with axis horizontal.* Fig. 501.



$$\left.\begin{array}{l}\text{vol. immers.}\\ = V\end{array}\right\} = [\text{area of segment } ABD] \times l$$

$$= \left(r^2\alpha - \tfrac{1}{2}r^2 \sin 2\alpha\right)l \quad \therefore \text{ since } V = \frac{G'}{\gamma}$$

$$lr^2\left[\alpha - \tfrac{1}{2}\sin 2\alpha\right] = \frac{G'}{\gamma} \quad \ldots\ldots (6)$$

Fig. 500. From this transcendental equation we can obtain α, by trial, in *radians*; (see *Example* in § 410). and finally h since $h = r(1 - \cos\alpha)$ .... (7)

Example 1. A sphere of 40 inches diameter is observed to have a depth of flotation $h = 9$ in. in water. Required its weight G'. From eq. (5) (inch, lb. sec.) we have

$$G' = [62.5 \div 1728]\,\pi\, 9^2\left[20 - \tfrac{1}{3}\times 9\right] = 156.5 \text{ lbs.}$$

The sphere may be hollow, e.g. of sheet metal loaded with shot, constructed in any way, so long as G' and the vol. V of displacement remain unchanged. But if the sphere is *homogeneous* its *heaviness* (§ 7) γ' must be $= G' \div V' = G' \div \frac{4}{3}\pi r^3$ $= (156.5) \div \frac{4}{3}\pi\, 20^3 = .00466$ lbs. per cubic inch, and hence referred to water, its *specif. grav.* is σ = about 0.13

Example 2. The right cylinder in Fig. 501 is homogeneous and 10 inches in diameter, and has a specif. grav. (referred to water) of $\sigma = 0.30$. Required the depth of flotation h.

Its heaviness must be $\gamma' = \sigma\gamma$: its weight $G' = V'\sigma\gamma = \pi r^2 l\sigma\gamma$

$\therefore$ [eq.(6)], $r^2 l\left[\alpha - \frac{1}{2}\sin 2\alpha\right] = \pi r^2 l\sigma$, $\therefore$ $\alpha - \frac{1}{2}\sin 2\alpha = \pi\sigma$ (involving abstract numbers only). Trying $\alpha = 60°$ ($= \frac{1}{3}\pi$ in radians) we have $\frac{1}{3}\pi - \frac{1}{2}\sin 120° = 0.614$, whereas $\pi\sigma = .9428$

For $\alpha = 70°$, $1.2217 - \frac{1}{2}\sin 140° = 0.9003$

For $\alpha = 71°$, $1.2391 - \frac{1}{2}\sin 142° = 0.9313$

For $\alpha = 71°\,22'$, $1.2455 - \frac{1}{2}\sin 142°\,44' = 0.9428$ which may be considered sufficiently close. Now from eq.(7),

$$h = (5\text{ in.})(1 - \cos.\,71°\,22') = 3.40\text{ in.}\quad Ans.$$

429. DRAUGHT OF SHIPS. In designing a ship, especially if of a new model, the position of the centre of gravity is found by eq.(3) of § 23 (with weights instead of volumes); i.e., the sum of the products obtained by multiplying the weight of each portion of the hull and cargo by the distance of its centre of gravity from a convenient reference-plane (e.g. the horizont. plane of the keel bottom) is divided by the sum of the weights, and the quotient is the distance of the centre of grav. of the whole from the reference plane. Similarly the distance from another reference plane is determined. These two co-ordinates and the fact that the cent. of grav. lies in the median vertical plane of symmetry of the ship (assuming a symmetrical arrangement of the framework and cargo) fix its location. The total weight, G', = of course the sum of the individual weights just mentioned. The *centre of buoyancy* for *any assumed draught* and corresponding position of ship is found by the same method; but more simply, since it is the centre of gravity of the imaginary homogeneous volume between

the water-line plane and the wetted surface of the hull. This volume (of "displacement") is divided into an even number (say 4 to 8) of horizontal laminae of equal thickness and Simpson's Rule applied to find the volume (i.e. V in preceding formulae) and also (eq. 3 § 23) the height of its cent. of grav. above the keel. Similarly, by division into (from 8 to 20) vertical slices, ⊥ to keel, (an even number and of equal thickness) we find the distance of the cent. of grav. from the bow. Thus the centre of buoyancy is fixed, and the corresponding buoyant effort $V\gamma$ (technically called the *displacement* and expressed in tons usually) computed, for any assumed draught of ship (upright). That position in which the "*displacement*" $= G' =$ weight of ship is the position of equilibrium of the ship when floating upright in still water, and the corresponding draught is noted. As to whether this equilibrium is stable or unstable, see next §.

In most ships the centre of grav. C, is several feet above the centre of buoyancy, B and a foot or two below the water line.

After a ship is afloat and its draught actually noted its total weight $G' = V\gamma$ can be computed, the values of V for different draughts having been previously calculated in advance. In this way the weights of different cargoes can also be measured.

Example. A ship having a *displacement* of 5000 tons is itself 5000 tons in weight and displaces a volume of *salt* water

$$V = G' \div \gamma = 10\,000\,000 \text{ lbs.} \div 64 \text{ lbs per cub ft.} = 156250 \text{ cub ft.}$$

430. ANGULAR STABILITY OF SHIPS.

If a vessel floating upright were of the peculiar form and position of Fig. 502 (water-line section having an area = zero) its tendency to regain that position, or depart from it, when slightly inclined an angle ϕ from the vertical is due to the action of the couple now formed by the equal ∥ forces $V\gamma$ and G', which are no longer directly opposed. This couple is called

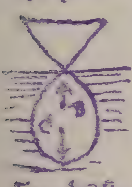


Fig. 502

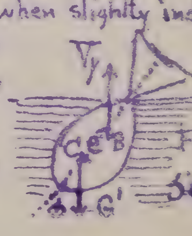


Fig. 503

a *righting couple* if it acts to restore the first position (as in Fig. 503 where C is lower than B) and an *upsetting couple* if the reverse, C above B. In either case the moment of the couple is $= V\gamma\ \overline{BC}\sin\phi = V\gamma\, e\sin\phi$, and *the centre of buoyancy B does not change its position in the vessel*, since the water-displacing shape remains the same, i.e., no new portions of the vessel are either immersed, or raised out of the water.

But in a vessel of ordinary form, when turned an angle ϕ from the vertical, Fig. 504, there is a *new* centre of buoyancy B_1, corresponding to the new shape A_1N_1D of the displacement-volume and the couple to right the vessel (or the reverse) consists of the two forces G' at C and $V\gamma$ at B_1, and has a moment (which we may call M, or *moment of stability*) of a value (§ 28)

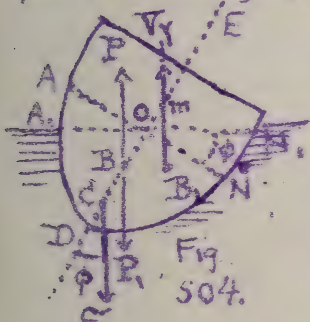


Fig. 504.

$$M = V\gamma\ \overline{mC}\sin\phi \quad \ldots\ldots (1)$$

Now conceive put in at B (centre of buoyancy of the upright position) two vertical and opposite forces, each $= V\gamma = G'$, calling them P and P_1 (see § 20), Fig. 504. We can now regard the couple $[G', V\gamma]$ as replaced by the two couples $[G', P]$ and $[P_1, V\gamma]$; for evidently $V\gamma\ \overline{mC}\sin\phi = V\gamma\ \overline{BC}\sin\phi + V\gamma\ \overline{mB}\sin\phi$ (§§ 33 and 34). $\therefore$

$$M = V\gamma\ \overline{BC}\sin\phi + V\gamma\ \overline{mB}\sin\phi \quad \ldots\ldots (2)$$

But the couple $[G', P]$ would be the only one to right the vessel if no new portions of the hull entered the water or emerged from it, in the inclined position; hence the other couple $[P_1, V\gamma]$ owes its existence to the emersion of the wedge AOA_1, and the immersion of the wedge NON_1, i.e., to the loss of a buoyant force $Q = (\text{vol. } AOA_1)\gamma$ on one side and the gain of an equal buoyant force on the other; $\therefore$ this couple $[P_1, V\gamma]$ is the equivalent of the couple $[Q, Q]$ Fig. 505 formed by

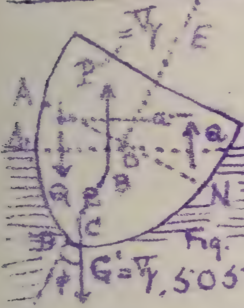


Fig. 505

putting in at the cent. of buoyancy of each of the two wedges a vertical force

$Q = (\text{vol. of wedge})\gamma = V_w \gamma$. (see fig.)

If a denotes the arm of this couple we may write

$$V\gamma \; \overline{mB} \sin\phi \; [\text{of eq.}(2)] = V_w \gamma a \; \ldots (3)$$

and hence, denoting $\overline{BC}$ by e, we have

$$M = \pm V\gamma e \sin\phi + V_w \gamma a \; \ldots\ldots (4)$$

the negative sign in which is to be used when C is above B (as with most ships). O, the intersection of ED and AN, does not necessarily lie on the new water-line plane A_1N_1.

Example. If a ship of ($V\gamma =$) 3000 tons displacement with C 4 ft. above B (i.e. $e = -4$ ft), is deviated 10° from the vertical, in salt water, for which angle the wedges AOA_1 and NON_1 have each a volume of 4000 cub. feet, while the horizontal distance a between their centres of buoyancy is 18 feet, the moment of the acting couple will be, from eq.(4) (ft. ton. sec. system, in which γ of salt water $= 0.032$)

$$M = -3000 \times 4 \times 0.1736 + 4000 \times 0.032 \times 18 = 220.8 \text{ ft. tons}$$

which being + indicates a *righting* couple.

REMARK. If with a given ship and cargo this moment of stability, M, be computed, by eq.(4), for a number of values of ϕ, and the results plotted as ordinates (to scale) of a curve,

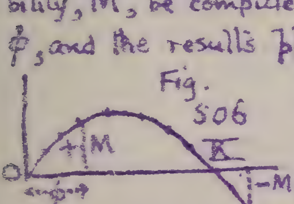


Fig. 506

ϕ being the abscissa, the curve obtained is indicative of the general stability of the ship. For some value of $\phi = OK$, (as well as for $\phi = 0$) the value of M is zero, and for $\phi > OK$ M is negative, indicating an *upsetting couple*. That is for $\phi = 0$ the equilibrium is stable but for $\phi = OK$, unstable; and $M = 0$ in both positions. From eq.(4) we see why, if C is above B, instability does not necessarily follow.

431. METACENTRE OF A SHIP. Referring again to Fig. 504, we note that the entire couple $[G', V_\gamma]$ will be a righting couple or an upsetting couple according as the point m (the intersection of the vertical through B_1, the new centre of buoyancy, with BC prolonged) is above or below the centre of gravity C of the ship. The location of this point m changes with ϕ, but as ϕ becomes very small (and ultimately zero) m approaches a *definite position* on the line DE, though not occupying till $\phi = 0$. This limiting position of m is called the *metacentre*, and accordingly the following may be stated:

A ship floating upright is in stable equilibrium if its metacentre is above its centre of gravity; and vice versâ. In other words for a slight inclination from the vertical a righting, and not an upsetting, couple is called into action if m is above C. To find the metacentre, by means of the distance $\overline{Bm}$, we have, from eq. (3)

$$\overline{mB} = \frac{V_w \gamma\, a}{V_\gamma \sin\phi} \quad \ldots\ldots (5)$$

and ultimately make $\phi = 0$. Now the moment $(V_w \gamma)a$ = the sum of the moments about the horizontal fore-and-aft water-line axis OL, Fig. 507, of the buoyant efforts due to the immersion of the separate vertical elementary prisms of the wedge OLN_1N, minus those lost, from emersion, in the wedge OLA_1A. Let OA_1LN_1 be the new water-line section of the ship when inclined a small angle ϕ from the vertical ($\phi = NON_1$), OALN the old water-line. Let z = the ⊥ distance of any elementary area dF of the water-line section from OL (which is the intersection of the two water-line planes). Each dF is the base of an

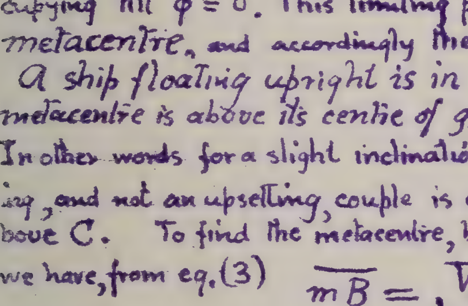


Fig. 507

elementary prism, with altitude $= \phi z$, of the wedge N_1OLN (or of wedge A_1OLA when z is negative). The buoyant effort of this prism $=$ (its vol.) $\times \gamma = \gamma z \phi dF$, and its moment about OL is $\phi \gamma z^2 dF$. Hence the total moment, $= Qa$, or $V_w \gamma a$, of Fig. 505, $= \phi\gamma \int z^2 dF = \gamma\phi \times I_{OL}$ of water-line section

in which I_{OL} denotes the "mom. of inertia" (§ 85) of the plane figure OALNO, about the axis OL. Hence from (5) putting $\phi = \sin\phi$ (true when $\phi = 0$) we have $\overline{mB} = I_{OL} \div V$

and $\therefore$ the distance $\overline{mC}$, of the metacentre, m, above C, the cent. of gravity of ship, Fig 504

$$\left.\begin{array}{l}\text{and } \therefore \text{ the distance } \overline{mC}, \text{ of} \\ \text{the metacentre } m \text{ above C, the cent.} \\ \text{of gravity of ship, Fig 504}\end{array}\right\} = \frac{I_{OL} \text{ of wat.-line sec.}}{V} \pm e \quad \dots (6)$$

in which $e = BC =$ distance from cent. of grav. to the cent. of buoyancy, the negative sign being used when C is above B, and $V =$ whole volume of water displaced by the ship. We may also write, from eqs. (6) and (1), for small values of ϕ,

$$\text{Mom. of righting couple} = M = V\gamma \sin\phi \left[\frac{I_{OL}}{V} \pm e\right] \dots (7)$$

or

$$M = \gamma \sin\phi \left[I_{OL} \pm Ve\right] \dots\dots$$

Eq.(7) will give near approximations for $\phi < 10$ or $15°$ with ships of ordinary forms.

Example 1. A homogeneous right parallelopiped, of heaviness γ', floats upright as in Fig. 508. Find its metacentre and whether the equilibrium is stable.

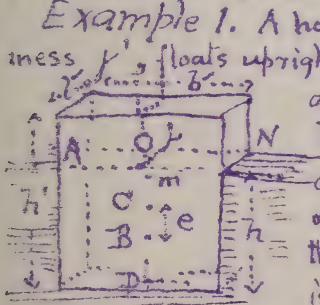


Fig. 508.

Here the centre of gravity, C, being the centre of figure, is of course above B, the cent. of buoyancy; $\therefore$ e is *negative*. B is the centre of grav. of the displacement, and is $\therefore$ a distance $\frac{1}{2}h$ below the waterline. From eq.(2) § 428, $h = h'\gamma' \div \gamma$ and since $CD = \frac{1}{2}h'$, and $BD = \frac{1}{2}h$, $\therefore e = \frac{1}{2}(h'-h)$. i.e. $e = \frac{1}{2}h'\left[1 - \frac{\gamma'}{\gamma}\right]$; while, § 90, I_{OL} of water-line section $AN = \frac{1}{12} l' b'^3$. Also $V = b'h\,l' = b'l'h'\frac{\gamma'}{\gamma}$.

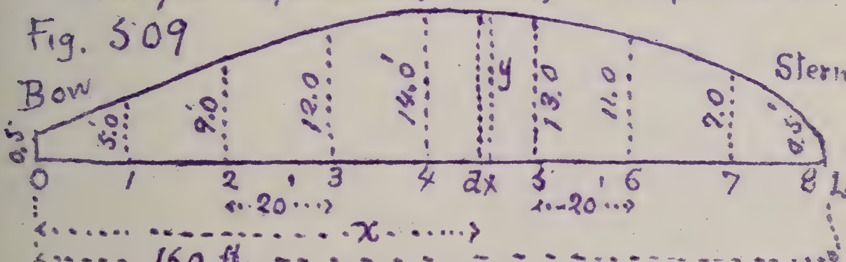
∴ from eq. 6 $\Big\}\ \overline{mC} = \dfrac{l'b'^3\gamma}{12\cdot b'h'l'\gamma'} - \dfrac{1}{2}h'\left[1-\dfrac{\gamma'}{\gamma}\right] = \dfrac{\gamma}{12h'\gamma'}\left[b'^2 - 6h'^2\dfrac{\gamma'}{\gamma}\left(1-\dfrac{\gamma'}{\gamma}\right)\right]$

Hence if b'^2 is $> 6h'^2\dfrac{\gamma'}{\gamma}\left(1-\dfrac{\gamma'}{\gamma}\right)$ the position in Fig. 508 is one of stable equil., and vice versâ. E.g., if $\gamma' = \frac{1}{2}\gamma$, $b' = 12^{in}$ and $h' = 6$ in., we have in.-lb.-sec. $\Big\}\ \overline{mC} = \dfrac{1}{36}\left[144 - 6\times\dfrac{36}{2}\left(1-\dfrac{1}{2}\right)\right] = 2.5$ inches

The equil. will be unstable if, with $\gamma' = \frac{1}{2}\gamma$, b' is made < than $1.225h'$, for putting $\overline{mC} = 0$ we obtain $b' = 1.225h'$

Example 2. (ft. lb. sec.) Let Fig. 509 represent the

Fig. 509



half water-line section of a loaded ship of $G' = V\gamma = 800$ tons displacement, required the height of the metacentre above the centre of buoyancy, i.e. $\overline{mB} = ?$ (See eq. just before eq. (6).)

Now the quantity I_{OL} of the water line section may, from symmetry (see § 93) be written $I_{OL} = 2\int_0^L \frac{1}{3}y^3\,dx$, in which y = the ordinate ⊥ to the axis OL at any point, and this, again, by Simpson's Rule for approx. integration, OL being divided into an even number n of equal parts and ordinates erected (see Figure), may be written

$$I_{OL} = \frac{2}{3}\cdot\frac{\overline{OL}-0}{n}\left[y_0^3 + 4(y_1^3 + y_3^3 + \dots) + 2(y_2^3 + y_4^3 + \dots \text{not last one.}) + y_n^3\right]$$

From which, by numerical substitution, see figure for dimensions, $n = 8$,

$$I_{OL} = \frac{2}{3}\cdot\frac{160}{3\times 8}\left[(0.5)^3 + 4(5^3 + 12^3 + 13^3 + 7^3) + 2(9^3 + 14^3 + 11^3) + 0.5^3\right]$$

$$\begin{array}{lcc} & 125 & 729 \\ \text{or,} & 1728 & 2744 \\ \text{(approx. of course)} & 2197 & 1331 \\ & 343 & \end{array}$$

$$I_{OL} = \frac{40}{9}\left[0.125 + 4 \times 4393 + 2 \times 4804 + 0.125\right]$$

$$= 120801 \text{ biquad. ft.} \;\therefore\; \overline{mB} = \frac{I_{OL}}{V} = \frac{120801}{[2020000 \div 64]}$$

$$= 3.8 \text{ feet}$$

That is, the metacentre is 3.8 ft. above the centre of buoyancy; and hence, if BC = 2 feet, is 1.90 ft. above the cent. of grav. [See Johnson's Cyclopaedia, article *Naval Architecture*]

Chap. III. Hydrostatics (continued); Gaseous Fluids.

432. THERMOMETERS. The temperature, or "*hotness*," of liquids has, within certain limits, but little influence on their statical behavior, but with gases must always be taken into account, since the three quantities, *tension*, *temperature*, and *volume* of a given mass of gas are connected by a nearly invariable law; as will be seen.

An *air-thermometer*, *Fig.* 510, consists of a large glass bulb filled with air, from which projects a fine straight tube of even bore (so that equal lengths represent equal volumes). A small drop of liquid, A, separates the internal from the external air, both of which are at a tension of one atmosphere (14.7 lbs. per sq. inch). When the bulb is placed in melting ice (freezing point) the drop stands at some point F in the tube; when in boiling water, the drop is found at B on account of the expansion of the internal air under the influence of the heat imparted to it. (The glass also expands, but only about 1 : 150 th as much; this will be neglected). The distance FB

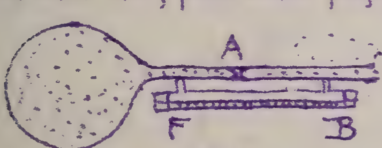


Fig. 510

along the tube may now be divided into a convenient number of equal parts called **degrees**. If into one hundred degrees, it is found that each degree represents a volume equal to the $\frac{367}{100000}$ (.00367) part of the total volume occupied by the air at freezing point, whatever the constant pressure may have been; i.e., the increase in volume from freezing to boiling point is .0.367 of the volume at freezing, *the pressure remaining constant*, (and having any value whatever, within ordinary limits, so long as it is the same both at freezing and boiling.)

As it not always practicable to preserve the pressure constant under all circumstances with an air-thermometer, we use the common mercurial thermometer for most practical purposes. In this, the tube is sealed at the outer extremity, with a vacuum above the column of mercury, and gives indications which agree very closely with those of the air-thermometer. That equal absolute increments of volume should imply equal increments of heat imparted to these thermometric fluids (under constant pressure) could not reasonably be asserted without satisfactory experimental evidence. This, however, is not altogether wanting, so that we are enabled to say that within a moderate range of temperature equal increments of heat produce equal increments of volume in a given mass not only ~~[illegible]~~ of atmospheric air but of the so called "perfect" or "permanent" gases, oxygen, nitrogen, hydrogen, etc. (so named before it was found that they could be liquefied). This is nearly true for mercury and alcohol, as well, but not for water.

The scale of a mercurial thermometer is fixed; but with an air-thermometer, we should have to use a new scale, and in a new position on the tube, for each value of the pressure.

433. THERMOMETRIC SCALES. In the *Fahrenheit* scale the tube between freezing and boiling is marked off into 180 equal parts and the zero placed at 32 of these parts below the freezing point, which is ∴ +32°, and the boiling point +212°.

The *Centigrade*, or *Celsius*, scale which is the one chiefly used in scientific practice, places its zero at freezing and 100°.

at boiling point. Hence to reduce

Fahr. readings to Centigrade, subtract 32° and X by $5/9$

Cent. " " " Fahrenheit, mult. by $9/5$ and add 32°.

434. ABSOLUTE TEMPERATURE. Experiment also shows that if a mass of air or other perfect gas is confined in a vessel whose volume is but slightly affected by changes of temperature, equal increments of temperature (and therefore equal increments of heat imparted to the gas) produce equal increments of tension (i.e. pressure per unit area); or, as to the amount of the increase, that when the temperature is raised by an amount 1° Cent., the tension is increased $\frac{1}{273}$ of its value at freezing point. Hence, theoretically, a barometer (containing a liquid unaffected by changes of temperature) communicating with the confined gas (whose volume remains constant) would by its indications serve as a thermometer, Fig. 511, and the attached scale could be graduated accordingly. Thus, if the column stood at A when the temperature was freezing, A would be marked 0° on the Centigrade system, and the degree spaces above and below A would each $= \frac{1}{273}$ of the height AB, and ∴ the point B (cistern level) to which the column would sink if the gas-tension were zero would be marked −273° Cent.

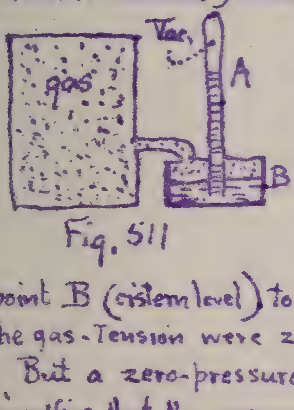


Fig. 511

But a zero-pressure, in the Kinetic Theory of gases (§ 393) signifies that the gaseous molecules, no longer impinging against the vessel walls (so that the press. = 0) have become motionless; and this, in the Mechanical Theory of Heat, or Thermodynamics, implies that *the gas is totally destitute of heat*. Hence this ideal temperature of −273° Cent., or −461° Fahr., is called the *Absolute Zero of Temperature*, and by reckoning temperatures from it as a starting point, our formulae will be rendered much more simple and compact. Temperature so reckoned is called *absolute temperature* and will be denoted by

the letter T. Hence the following rules for reduction:
Absol. temp T in Cent. degrees = Ordinary Cent. + 273°
Absol. Temp. T in Fahr. degrees = Ordinary Fahr. + 461°
For Example, for 20° Cent., T = 293° Abs. Cent.

435. DISTINCTION BETWEEN GASES AND VAPORS.

All known gases can be converted into liquids by a sufficient reduction of temperature or increase of pressure, or both; some, however, with great difficulty, such as atmospheric air, oxygen, hydrogen, nitrogen, etc., having been only recently (1878) reduced to the liquid form. A *vapor* is a gas near the point of liquefaction and does not show that regularity of behavior under changes of temperature and pressure characteristic of a gas when much above the point of liquefaction. All gases treated in this chapter (except steam) are supposed in a condition far removed from this stage. The following will illustrate the properties of vapors:

Fig. 512. Let a quantity of liquid, say water, be introduced into a closed vacuous space of considerably larger volume than the water and furnished with a manometer and thermometer. Vapor of water immediately begins to form in the space above the liquid and continues to do so until its pressure attains a definite value dependent on the temperature; e.g. if the temp. is 70° Fahr. the vapor ceases to form when the tension reaches a value of 0.36 lbs. per sq. inch. If

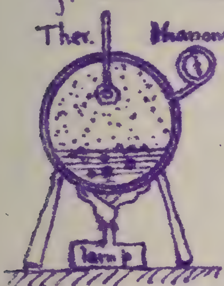


Fig. 512.

heat be gradually applied to raise the temperature, more vapor will form (with ebullition, i.e. from the body of the liquid, unless the heat is applied very slowly) but the tension will not rise above a *fixed value for each temperature* independent of size of vessel so long as there is any *liquid left*. Some of these corresponding values, for water, are as follows: for a

Fahr. temp =	70°	100°	150°	212°	220°	287°	300°
tens., lbs. per sq. in =	0.36	0.93	3.69	14.7 = one atm.	17.2	55.0	67.2

At any such stage the vapor is said to be *saturated*.

Finally, at some temperature, dependent on the ratio of the original volume of water to that of the vessel, all of the water will have been converted into vapor (i.e. steam), and if the temperature be still further increased the tension also increases, but no longer depends on the temperature alone but also on the heaviness of the vapor when the water disappeared. The vapor is now said to be *superheated*, and conforms more in its properties to a "perfect gas".

From some experiments there seems to be reason to believe that at a certain temperature, called the *critical temperature*, different for different liquids, all of the liquid in the vessel (if any remains, and supposing the vessel strong enough to resist the pressure) is converted into vapor, whatever be the size of the vessel. That is, above the critical temperature the substance is necessarily gaseous, in the most exclusive sense, incapable of liquefaction by pressure alone, while below this temperature it is a vapor, and liquefaction will begin if, by compression in a cylinder and consequent increase of pressure the tension can be raised to the value corresponding for a state of saturation to the temperature (in such a table as that just given for water). For example if vapor of water at 220° Fahr., and a tension of 10 lbs. per sq. in, (this is superheated steam, since 220° is higher than the temp. which for saturation corresponds to $p = 10$ lbs per sq.in.) is compressed slowly (slowly, to avoid change of temperature) till the tension rises to 17.2 lbs per sq. in, which (see table above) is the press. of saturation for a temp. of 220° Fahr. for water-vapor, the vapor is saturated, i.e., liquefaction is ready to begin and during any further slow reduction of volume the press. remains constant and some of the vapor is liquefied.

By "perfect gases" or gases proper we may understand, i.e., those which can not be liquefied by pressure without great reduction of temperature, i.e., whose "critical

temperature" is very low.

436. LAW OF CHARLES (AND OF GAY LUSSAC) The mode of graduation of the air thermometer may be expressed in the following formula, which holds good within the ordinary limits of experiment for a given mass of *any perfect gas*, the tension remaining constant,

$$V = V_0 + 0.00367\, V_0 t = V_0 (1 + .00367\, t) \ldots\ldots (1)$$

in which V_0 denotes the volume occupied by the given mass at freezing point, under the given pressure, V its volume at any other temperature t Centigrade under the same tension. Now 273 being the reciprocal of .00367 we may write

$$V = V_0 \frac{(273 + t)}{273} \text{ i.e. } \frac{V}{V_0} = \frac{T}{T_0} \ldots\ldots \left.\begin{matrix}\text{press.}\\ \text{const.}\end{matrix}\right\} \ldots\ldots (2)$$

(see § 434) in which T_0 = the absolute temperature of freezing point = 273° Abs. Cent., and T the absol. temp. corresponding to t Cent. Eq. (2) is also true when T and T_0 are both expressed in Fahr. degrees (from abs. zero, of course). Accordingly we may say that, *the pressure remaining the same, the volume of a given mass of gas varies directly as the absol. temperature.*

Since the weight of the given mass of gas is invariable, at a given place on the earth's surface, we may

always use the equat. $V\gamma = V_0\gamma_0 \ldots\ldots\ldots (3)$

pressure constant, or not, and hence (2) may be rewritten

$$\frac{\gamma_0}{\gamma} = \frac{T}{T_0} \ldots\ldots \text{ press. const} \ldots\ldots\ldots (4)$$

i.e., *if the press. is constant, the heaviness (and ∴ the specific gravity) varies inversely as the absolute temperature.*

Experiment also shows, § 434, that if the volume (and ∴ the heaviness, eq. 3) remains constant, while the temperature varies, the tension p will change according to the following relation, in which p_0 = the tension when the temp. is freezing,:

$$p = p_0 + \frac{1}{273} p_0 t = p_0 \frac{273+t}{273} \quad \ldots\ldots\ldots (5)$$

t denoting the Cent. temp. $\therefore$ transforming, as before, we have $\Big\}$ $$\frac{p}{p_0} = \frac{T}{T_0} \quad \ldots\ldots \begin{Bmatrix}\text{vol. and } \therefore \\ \text{heav. const.}\end{Bmatrix} \ldots\ldots (6)$$

or, the volume and heaviness remaining constant, the tension of a given mass varies directly as the absolute temperature. This is called the LAW OF CHARLES (or of GAY LUSSAC)

437. GENERAL FORMULAE FOR ANY CHANGE OF STATE OF A PERFECT GAS. If any two of the three quantities, viz: *volume* (or heaviness), *tension*, and *temperature*, are changed, the value of the third is determinate from those two according to a relation proved as follows: (remembering that henceforth the absolute temperature only will be used, T, § 434) Fig. 512 a.

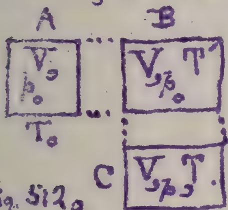


Fig. 512 a

At A a certain mass of gas at a tension of p_0, one atmosphere, and absolute temp. T_0, freezing, occupies a volume V_0. Let it now be heated to an absol. temp $= T'$, without change of tension (expanding behind a piston for instance) and its volume will increase to a value V which from (2) § 436, will satisfy the relation $\Big\}$ $$\ldots \frac{V}{V_0} = \frac{T'}{T_0} \quad \ldots\ldots\ldots (7) \quad \text{(B in figure)}$$

Let it now be heated without change of volume to an absol. temp. T (C in figure). Its volume is still V, but the tension has risen to a value p, such that comparing B and C, eq (6) $\Big\}$ $$\frac{p}{p_0} = \frac{T}{T'} \quad \ldots\ldots (8)$$

Combining (7) and (8) we obtain for any state, in which the tension is p, volume V, and absol. temp. T, in

$$\text{GENERAL} \ldots \frac{pV}{T} = \frac{p_0 V_0}{T_0}; \text{ or } \frac{pV}{T} = \text{a constant} \ldots\ldots (9)$$

$$\text{or, GENERAL} \ldots \frac{p_m V_m}{T_m} = \frac{p_n V_n}{T_n} \quad \ldots\ldots (10)$$

which since

$$(\text{GENERAL})\ldots\ V\gamma = V_0\gamma_0 = V_m\gamma_m = V_n\gamma_n \ldots\ldots\ldots(11)$$

is true for any change of state, we may also write

$$(\text{GENERAL})\ldots\ \frac{p}{\gamma T} = \frac{p_0}{\gamma_0 T_0}\ ;\ldots\ldots(12);\ \text{or,}\ \frac{p_m}{\gamma_m T_m} = \frac{p_n}{\gamma_n T_n}\ \ldots(13)$$

These equations, (9) to (13) inclus., hold good for any state of a mass of any perfect gas (most accurately for air). The subscript 0 refers to the state of one atmos. tension and freezing point temperature; m and n to any two states whatever (within practical limits); γ is the heaviness, §§ 7 and 394, and T the absolute temperature, § 434.

438 EXAMPLES. *Example 1.* What cubic space will be occupied by 2 lbs. of hydrogen gas at a tension of two atmos. and a temperature of 27° Cent.? With the *inch-lb.-sec.* system we have $p_0 = 14.7$ lbs. per sq. inch, $\gamma_0 = [.0056 \div 1728]$ lbs. per cubic , and $T_0 = 273°$ Abs. Cent., when the gas is at freezing point at one atmos. (i.e. in state sub-zero). In the state mentioned in the problem we have $p = 2 \times 14.7$ lbs. per sq. in., $T = 273 + 27 = 300°$ Abs. Cent., while γ is required. Hence from eq. (12) we have

$$\ldots\ \frac{2 \times 14.7}{\gamma\ 300} = \frac{14.7}{(.0056 \div 1728)\,273}\ \therefore \gamma = \frac{.0102}{1728}$$

lbs. per cub. in. = .0102 lbs. per cub. foot; and if the total weight $G = V\gamma$ is to be 2 lbs. we have (ft. lb. sec.) $V = 2 \div .0102 = 14.6$ cub. feet. *Ans.*

Example 2. A mass of air originally at 24° Cent. and a tension indicated by a barometric column of 40 in. of mercury, has been simultaneously reduced to half its former volume and heated to 100° Cent.; required its tension in this new state, which we call the state n, m being the original state. Use the *inch-lb.-sec.* We have given, therefore $p_m = \frac{40}{30}\,14.7$ lbs per sq. inch, $T_m = 273 + 24 = 297°$ Abs. Cent., the ra-

to $V_m : V_n = 2:1$, and $T_n = 273 + 100 = 373°$ Abs. Cent. while p_n is the unknown quantity. From eq. (10) ∴

$$p_n = \frac{V_m}{V_n} \cdot \frac{T_n}{T_m} \cdot p_m = 2 \times \frac{373}{297} \cdot \frac{40}{30} \times 14.7 = 49.22 \text{ lbs. per}$$

sq. inch, which an ordinary steam-gauge would indicate as $(49.22 - 14.7) = 32.52$ lbs. per sq. inch.

Example 3. A mass of air, Fig. 513, occupies a rigid closed vessel at a temp. of 15° Cent. (= that of surrounding objects) and a tension of four atmos, [state m]. By opening a stop-cock a few seconds, and thus allowing a portion of the gas to escape quickly, and then shutting it, the remainder of the air [now in state n] is found to have a tension of only 2.5 atmos. (measured immediately); its temp. can not be measured immediately, for obvious reasons, and is less than before. To compute this temp., T_n, we allow the air now in the vessel to come again to the same temp. as surrounding objects, (15° Cent.) and note the tension to be 2.92 atmos. Call this last state, state r. Inch, lb. sec. The problem then stands thus:

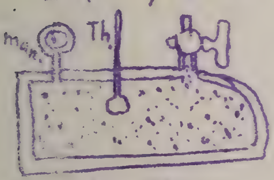


Fig. 513

$p_m = 4 \times 14.7$	$p_n = 2.5 \times 14.7$	$p_r = 2.92 \times 14.7$
$v_m = ?$	$v_n = ?$	$v_r = v_n$ since $V_r = V_n$
$T_m = 288°$ Abs. Cent.	$T_n = ?$ {principal unknown}	$T_r = T_m = 288°$ Ab. C.

From eq. 13, § 437: $\dfrac{p_m}{v_m T_m} = \dfrac{p_n}{v_n T_n}$ .....(1) [∴ noting that $v_n = v_r$] ... $\dfrac{p_n}{T_n} = \dfrac{p_r}{T_r}$ ....(2)

From (2) we have ... $$T_n = \frac{2.5 \times 14.7}{2.92 \times 14.7} \times 288 = 246° \text{ Ab. C.} = -27° \text{C.}$$

considerably below freezing, as a result of allowing the sudden escape of a portion of the air, and the consequent sudden expansion, and reduction of heaviness, of the remainder. In passing suddenly from state m to state n, this remainder altered

its heaviness (and its volume in inverse ratio) in the ratio [see (1)]

$$\frac{\gamma_n}{\gamma_m} = \frac{V_m}{V_n} = \frac{p_n}{p_m}\cdot\frac{T_m}{T_n} = \frac{2.5\times 14.7}{4\times 14.7}\cdot\frac{288}{246} = 0.73$$

Now the heaviness in state m (see eq. 12. § 437, also § 394)

was $\gamma_m = \frac{p_m}{T_m}\cdot\frac{\gamma_0 T_0}{p_0} = \frac{4\times 14.7}{288}\cdot\frac{.0807}{1728}\cdot\frac{273}{14.7} = \frac{.306}{1728}$

lbs per cub. in. $=$ .306 lbs per cub. ft. $\therefore \gamma_n = 0.73\times\gamma_m = 0.22$ lbs per cub. ft., and also, since $V_m = 0.73\, V_n$, about $\frac{27}{100}$ of the original quantity of air in vessel has escaped.

[*Note.* By numerous experiments like this, the law of cooling when a mass of gas is allowing to expand suddenly (as, e.g., behind a piston, doing work) has been determined; (and vice versâ, the law of heating under sudden compression); see § 442]

439. THE CLOSED AIR MANOMETER. If a manometer be formed of a straight tube of glass, of uniform cylindrical bore, filled with mercury and inverted in a cistern of mercury, a quantity of air having been left between the mercury and the upper end of the tube, which is closed, the tension of the air (known by observing its volume and temperature) must be added to that due to the mercury column to obtain the tension p' to be measured. The advantage of this kind of instrument is, that to measure great tensions the tube need not be very long.

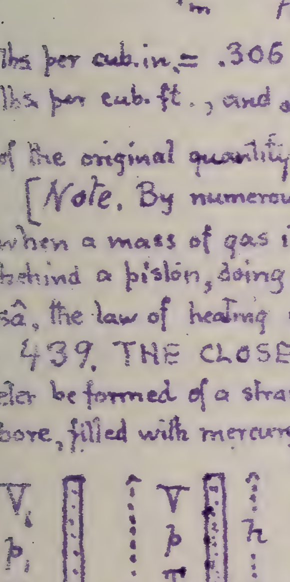


Fig. 514

Fig. 514. Let the temp. T_1 of whole instrument, and the tension p_1 of the air or gas in the cistern be observed when the mercury in the tube stands at the same level as that in the cistern. The tension of the air in the tube must also be p_1, its temp. T_1, and its volume

is $V_1 = Fh_1$, F being the sectional area of the bore of the tube; see figure. Gas of unknown tension p' being admitted to the cistern, the temperature of the whole instrument being $= T$, the heights h and h'' are observed ($h + h''$ does not $= h_1$ unless the cistern is very large) and p' computed as follows

$$p' = h''\gamma_m + p \text{ - - - - - - - - } \quad (1)$$

in which $p =$ the tension of the air in the tube, and γ_m the heaviness of mercury. But from eq. (10) § 437, putting $V_1 = Fh_1$ and $V = Fh$

$$p = p_1 \frac{V_1}{V} \cdot \frac{T}{T_1} = \frac{h_1}{h} \cdot \frac{T}{T_1} \cdot p_1 \text{ . . . . } (2)$$

Hence finally from (1) and (2)

$$p' = h''\gamma_m + \frac{h_1}{h} \cdot \frac{T}{T_1}\, p_1 \text{ . . . . } (3)$$

Since T_1, p_1, and h_1 are fixed constants for each instrument we may, from (3), compute p' for any observed values of h and T, (N.B. T and T_1 are absolute temperatures), and construct a series of tables each of which shall give values of p' for any values of h, and one special value of T.

Example. Supposing the fixed constants of a closed air manometer to be (in inch lb. sec. sys.) $p_1 = 14.7$ (or one atmosphere), $T_1 = 285°$ Abs. Cent. (i.e. 12° Cent.) and $h_1 = 3'\,4'' = 40$ inches; required the tension in the cistern indicated by $h'' = 25$ inches and $h = 15$ inches, when the temp. is $-3°$ Cent, or $T = 270°$ Abs. Cent. For mercury $\gamma_m = [848.7 \div 1728]$ (§ 394), (though strictly it should be specially computed for the temperature, since it varies about 2 : 100 000 of itself for each Cent. degree). Hence, eq. 3,

$$p' = \frac{25 \times 848.7}{1728} + \frac{40}{15} \cdot \frac{270}{285} \times 14.7 = 12.26 + 37.13 = 49.39$$

lbs. per sq. inch, or nearly 3½ atmos. [Steam-gauge would read 34.7]

440. MARIOTTE'S LAW, (or Boyle's,) TEMPERATURE CONSTANT, i.e., ISOTHERMAL CHANGE. If a mass of gas be compressed, or allowed to expand, *isothermally*, i.e.

without change of temperature (practically this can not be done unless the walls of the vessel are conductors of heat, and then the motion must be slow) eq. (10) of §437 now beomes (since $T_m = T_n$)

$$\left\{\begin{matrix}\text{MARIOTTE'S LAW}\\ \text{Temp. constant}\end{matrix}\right\} \ldots V_m p_m = V_n p_n\ ;\ \text{or}\ \frac{p_m}{p_n} = \frac{V_n}{V_m} \ldots\ldots (1)$$

i.e., *the temperature remaining unchanged, the tensions are inversely proportional to the volumes, of a given mass of a perfect gas.; or, the product of volume by tension is a constant quantity.* Again, since $V_m \gamma_m = V_n \gamma_n$, in any case,

$$\left\{\begin{matrix}\text{MARIOTTE'S LAW}\\ \text{Temp. constant}\end{matrix}\right\} \ldots \frac{p_m}{p_n} = \frac{\gamma_m}{\gamma_n} \ldots \text{or,}\ \frac{p_m}{\gamma_m} = \frac{p_n}{\gamma_n} \ldots (2)$$

i.e. *the pressures (or tensions) are directly proportional to the (first power of) the heavinesses, if the temp. is the same.*

This law which is very closely followed by all the perfect gases was discovered by Boyle in England and Mariotte in France more than two hundred years ago, but of course is only a particular case of the general formula, for any change of state, in §437. It may be verified experimentally in several ways. E.g. in Fig. 515, the tube OM being closed at the top, while PN is open, let mercury be poured in at P until it reaches the level A'B'. The air in OA' is now at a tension of one atmosphere. Let more mercury be slowly poured in at P, until the air confined in O has been compressed to a volume $OA'' = \frac{1}{2}$ of OA', and the height B''E'' measured; it will be found to be 30 inches, i.e. the tension of the air in O is now two atmospheres (corresponding to 60 in. of mercury). Again, compress the air in O to $1/3$ its original volume (when at one atmos.), i.e. to vol. $OA''' = 1/3\ OA'$, and the mercury height B'''E''' will be 60 inches, showing a tension of three atmos. in the air at O (90 inches of mercury in a barometer)

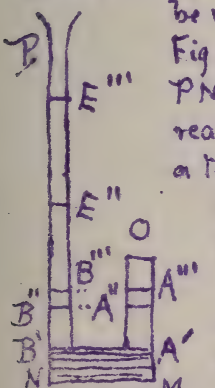


Fig. 515

It is understood that the temperature is the same, i.e., that time is given the compressed air to acquire the temperature of surrounding objects after being heated by the compression, if sudden.

Example 1. If a mass of compressed air expands in a cylinder behind a piston, having a tension of 60 lbs. per sq. inch (45.3 by steam gauge) at the beginning of the expansion, which is supposed slow (that the temperature may not fall); then when it has doubled in volume its tension will be only 30 lbs. sq. in.
" " " tripled " " " " " " " 20 " " "

and so on

[NOTE. The law of decrease of steam-pressure in a steam-engine cylinder, after the piston has passed the point of "cut-off", does not materially differ from Mariotte's law, which is often applied to the case of expanding steam; see § 443.]

Example 2. Diving Bell. Fig. 516. If the cylindrical diving-bell AB is 10 ft. in height, in what depth, $h = ?$, of salt water, can it be let down to the bottom, without allowing the water to rise in the bell more than $a = 4$ ft. Call the horiz. sectional area, F, the mass of air in the bell being constant, at a constant temperature.

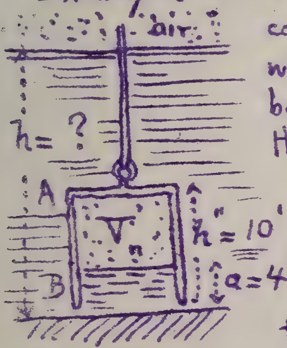


Fig. 516

First, algebraically; at the surface this mass of air occupied a volume $V_m = Fh''$ at a tension $p_m = 14.7 \times 144$ lbs. per sq. ft. while at the depth mentioned it is compressed to a volume $V_n = F(h'' - a)$ and is at a tension $p_n = p_m + (h-a)\gamma_w$ in which γ_w = heav. of salt water. Hence, from eq. (1),

$$p_m Fh'' = \left[p_m + (h-a)\gamma_w\right] F(h'' - a)\,; \therefore h = a\left[1 + \frac{p_m}{(h''-a)\gamma_w}\right]$$

and, numerically, ft. lb. sec.,

$$h = 4 \times \left[1 + \frac{14.7 \times 144}{(10-4) \times 64}\right] \therefore h = 26.05 \text{ feet.}$$

441. BAROMETRIC LEVELLING. By measuring with a barometer the tension of the atmosphere at two different levels, ~~simultaneously~~, and on a still day, the two localities not being too far removed horizontally, we may compute their vertical distance apart, if the temperature of the stratum of air between them is known, being the same, or nearly so, at both stations. Since the heaviness of the air is different in different layers of the vertical column between the two elevations N and M, Fig. 517, we can not immediately regard the whole of such a column as a free body (as was done with a liquid, § 397), but must consider a horizontal thin lamina, L, of thickness $= dz$ and at a distance $= z$ (variable) below M, the level of the upper station, N being the lower level at a distance, h, from M. The tension, p, must increase from M downwards since the lower laminae have

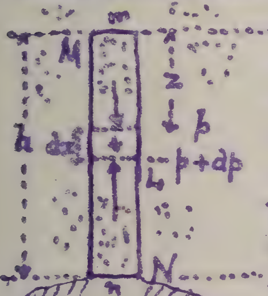


Fig. 517

to support greater weights of air above them, and, the temperature being hypothetically the same in all laminae, the heaviness, γ, must also increase, proportionally to p, from M downwards. Let the tension and heaviness of the air at the upper base of the lamina, L, be p and γ respectively. At the lower base, a distance dz below the upper, the tension is $p + dp$. Let the area of the base of lamina be F; then the vertical forces acting on the lamina are Fp downward, its weight $\gamma F dz$ downward, and $F(p+dp)$ upward. For its equil. Σ(vert. compons) must $= 0$

$$\therefore F(p+dp) - Fp - F\gamma dz = 0, \quad \text{i.e. } dp = \gamma dz \quad \ldots (1)$$

which contains three variables. But from Mariotte's law, § 440, if p_n and γ_n refer to the air at N, we may substitute $\gamma = \frac{\gamma_n}{p_n} p$ and obtain, after dividing by p, to separate the variables p and z,

$$\frac{p_n}{\gamma_n} \cdot \frac{dp}{p} = dz \quad \ldots\ldots\ldots (2)$$

Summing equations like

(2) one for each [illegible] between M (where $p = p_m$ and $z = 0$) and N (where $p = p_n$ and $z = h$) we have

$$\frac{p_n}{\gamma_n} \cdot \int_{p_m}^{p_n} \frac{dp}{p} = \int_0^h dz \text{ ; i.e. } h = \frac{p_n}{\gamma_n} \log_e \left[\frac{p_n}{p_m} \right] \ldots (3)$$

which gives h, the difference of level, or altitude, between M and N, in terms of the observed tensions p_n and p_m, and of γ_n the heaviness of the air at N, which may be computed from eq. (12) § 437, substituting from which we have finally

$$h = \frac{p_0}{\gamma_0} \cdot \frac{T_n}{T_0} \cdot \log_e \left[\frac{p_n}{p_m} \right] \ldots\ldots (4) \ldots \left\{ \begin{array}{l} \text{in which the sub-} \\ \text{script 0 refers to} \\ \text{freezing point and} \end{array} \right.$$

one atmos. tension; T_n and T_0 are absol. temps. For the ratio $p_n : p_m$ we may put the equal ratio $h_n : h_m$ of the actual barometric heights which measure the tensions. The $\log_e$ (or Naperian, or natural, or hyperbolic, log.) = (common log. to base 10) X 2.30258. From § 394, γ_0 of air = 0.08076 lbs. per cub. ft. and p_0 = 14.701 lbs. per sq. inch; $T_0 = 273°$ Abs. (Cent.)

If the temp. of the two stations (both in the shade) are not equal, a mean temp. = $\frac{1}{2}(T_m + T_n)$ may be used for T_n in eq. (4), for approx. results. Eq. (4) may then be written

$$(\text{ft. lb. sec. system}) \ldots h = 26213 \frac{T_n}{T_0} \cdot \log_e \left[\frac{p_n}{p_m} \right] \ldots \left\{ \begin{array}{c} \underline{\text{not}} \\ \underline{\text{homog.}} \end{array} \right\} (5)$$

The quantity $\frac{p_0}{\gamma_0}$ = 26213 ft., just substituted, is called the *height of the homogeneous atmosphere*, i.e., the ideal height which the atmosphere would have, if incompressible and non-expansive like a liquid, in order to exert a pressure of 14.701 lbs. per sq. inch upon its base, being throughout of a constant heaviness = .08076 lbs. per cub. foot.

By inversion of eq. (4) we may also write

$$p_m e^{\frac{\gamma_0}{p_0} \cdot \frac{T_0}{T_n} \cdot h} = p_n \ldots (6) \left\{ \begin{array}{l} \text{where } e = 2.71828 \\ \text{the Naperian base, to be} \\ \text{raised to the power} \end{array} \right.$$

indicated by the abstract number $\frac{\gamma_0}{p_0} \cdot \frac{T_0}{T_n} \cdot h$.

Example. Having observed as follows: (simultaneously)
At lower stat. N $h_n = 30.05$ in. mercury; temp. $= 77.6^\circ$ Fahr.
" upper " M $h_m = 23.66$ " " " " $= 70.4^\circ$ Fahr.
required the altitude h. From these figures we have a mean absol. temp of $461^\circ + \frac{1}{2}(77.6 + 70.4) = 535^\circ$ Abs. Fahr.
∴ from (5)

$$h = 26213 \times \frac{535}{493} \times 2.30258 \times \log_{10}\left[\frac{30.05}{23.66}\right] = 6800.6 \text{ feet.}$$

(Mt. Guanaruato, in Mexico, by Baron Von Humboldt.) Strictly we should take into account the latitude of the place since p_0 varies with g (see § 76) and also the decrease in the intensity of gravitation as we proceed further from the earth's centre.

442. ADIABATIC CHANGE. POISSON'S LAW. By an *adiabatic change of state*, on the part of a gas, is meant a compression or expansion in which work is done upon the gas (in compressing it) or by the gas (in expanding against a resistance) when there is *no transmission of heat* between the gas and enclosing vessel, or surrounding objects, by conduction or radiation. This occurs when the volume changes in a vessel of non-conducting material, or when the compression or expansion takes place so quickly that there is no time for transmission of heat to or from the gas.

The experimental facts are, that if a mass of gas in a cylinder be suddenly compressed to a smaller volume its temperature is raised, and tension increased more than the change of volume would call for by Mariotte's Law; and vice versâ, if a gas at high tension is allowed to expand in a cylinder and drive a piston against a resistance, its temperature falls, and its tension diminishes more rapidly than by Mariotte's Law.

Again, (see *Example* 3 § 438), if $^{27}/_{100}$ of the gas in a rigid vessel, originally at 4 atmos. tension and temp. of 15° Cent., is allowed to escape suddenly thro' a stop-cock into the outer air, the remainder while increasing its volume in the ratio 100 : 73 is found to have cooled to -27° Cent., and its tension to

have fallen to 2.5 atmos.; whereas by Mariotte's law if the temperature had been kept at 288° Abs. Cent. the tension would have been lowered to $\frac{73}{100}$ of 4, = 2.92 atmos., only.

The reason for this cooling during sudden expansion, is, according to the Kinetic Theory of Gases, that since the "sensible heat" (i.e. perceived by the thermometer) or "hotness" of a gas depends on the velocity of its incessantly moving molecules, ~~and that~~ and that each molecule after impact with a *receding* piston has a less velocity than before, the temperature necessarily falls; and *vice versâ*, when an advancing piston compresses the gas into a smaller volume.

If, however, a mass of gas expands without doing work, as when, in a vessel of two chambers, one a vacuum, the other full of air, communication is opened between them, and the air allowed to fill both chambers, no cooling is noted in the mass, *as a whole*.

By experiments similar to that in Example 3, § 438, it has been found that for air and the "perfect gases" in an adiabatic change of volume and (∴ of heaviness) the tension varies directly as the 1.41 power of the heaviness, and ∴ inversely as the same power of the volume. This is called *Poisson's Law*. For ordinary purposes (as Weisbach suggests) we may use $\frac{3}{2}$ instead of 1.41, and hence may write

$$\left.\begin{array}{l}\text{ADIABAT.}\\ \text{CHANGE}\end{array}\right\} \cdots \frac{p_m}{p_n} = \left(\frac{\gamma_m}{\gamma_n}\right)^{\frac{3}{2}} \cdots\cdots \text{or, } \frac{p_m}{p_n} = \left(\frac{V_n}{V_m}\right)^{\frac{3}{2}} \cdots\cdots (1)$$

and, combining this with the general eqs.ˢ 10 and 13 of § 437,

$$\left.\begin{array}{l}\text{ADIABAT.}\\ \text{CHANGE}\end{array}\right\} \cdots \frac{p_m}{p_n} = \left(\frac{T_m}{T_n}\right)^{3} \cdots\cdots\cdots (2)$$

and also

$$\left.\begin{array}{l}\text{ADIABAT.}\\ \text{CHANGE}\end{array}\right\} \cdots \frac{V_m}{V_n} = \left(\frac{T_n}{T_m}\right)^{2}; \text{ or, } \frac{\gamma_n}{\gamma_m} = \left(\frac{T_n}{T_m}\right)^{2} \cdots\cdots (3)$$

in which m and n refer to any two adiabatically related states.

T is the *absolute temperature*.

Example 1. Air in a cylinder at 20° Cent. is suddenly compressed to $1/6$ its original volume (and ∴ is six times as dense, i.e. has six times the heaviness, as before). To what temperature is it heated? Let m be the initial state, and n the final. From eq. (3) we have $\frac{T_n}{293} = \sqrt{\frac{6}{1}}$ ∴ $T_n = 718°$ Abs. C. or nearly double the absol. temp. of boiling water.

Example 2. After the air in Example 1. has been given time to cool again to 20° Cent. (temp. of surrounding objects) it is allowed to resume, suddenly, it first volume, i.e. to increase its volume sixfold by expanding behind a piston; to what temp. has it cooled. Here $T_m = 293°$ Abs. C., the ratio $V_m : V_n = 1/6$ and T_n is required. ∴ from (3)

$$\frac{T_n}{293} = \sqrt{1/6} \quad \therefore T_n = 293 \div \sqrt{6} = 119.5° \text{ Abs. Cent. or}$$

$-154°$ Cent., indicating extreme cold.

From these two examples the principle of one kind of ice-making apparatus is very evident. As to the work necessary to compress the air in Example 1, see § 447. It is also evident why motors using compressed air expansively have to encounter the difficulty of frozen watery vapor (present in the air to some extent.)

Example 3. What is the tension of the air in Example 1. (suddenly compressed to $1/6$ its original volume) immediately after the compression, if the original tension was one atmos.? That is, with $V_n : V_m = 1 : 6$, and $p_m = 14.7$ lbs. per sq. inch; $p_n = ?$

From eq. (1) (in. lb. sec.) $p_n = 14.7 \times 6^{3/2} = 14.7\sqrt{216} = 216$ lbs. per sq. inch, whereas if, after compression and without change of volume, it cools again to 20° Cent. the tension is only $14.7 \times 6 = 88.2$ lbs. per sq. inch. (now using Mariotte's law.)

443. WORK OF EXPANDING STEAM FOLLOWING MARIOTTE'S LAW. Although gases do not in general follow Mariotte's law in expanding behind a piston (without special provision for supplying heat) it is found that ~~steam (vapor)~~ the ~~mixture~~ of saturated steam (i.e. saturated at the beginning of the expansion) in a steam engine cylinder, being left to expand after the piston has passed the point of 'cut-off', diminishes very nearly in accordance with Mariotte's law, which may be applied in this case to find the work done per stroke.

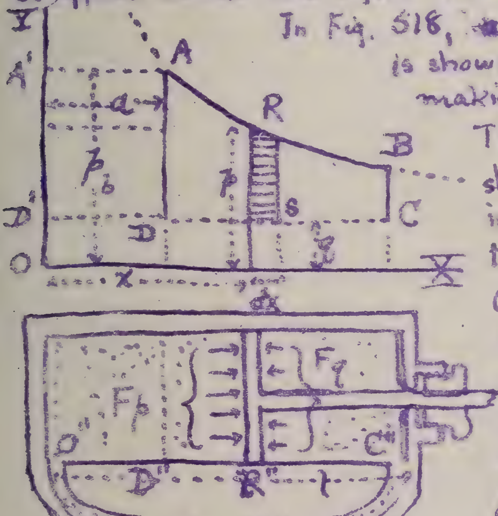


Fig. 518

In Fig. 518, a horizontal steam cylinder is shown in which the piston is making its left-to-right stroke. The "back pressure" is constant and $= Fq$, F being the area of the piston and q the intensity (i.e. per unit area) of the back- or exhaust-pressure; while the forward pressure on the left face of the piston $= Fp$, in which p is the steam pressure per unit area on the left, and is different at different points of the stroke. While the piston is passing from O'' to D'' p is constant being $= p_b$ the boiler pressure, since the steam port is still open. Between D'' and C'', however, the steam being cut off at D'', a distance a from O'', p decreases with Mariotte's law (nearly), and its value is $(Fa \div Fx)\,p_b$ at any point on C'' D'', x being the distance of the point from O''.

Above the cylinder, conceive to be drawn a diagram in which an axis OX is ∥ to the cylinder-axis, OY an axis is ⟂ to the same, while O is vertically above the left-hand

end of the cylinder. As the piston moves let the value of p corresponding to each of its positions be laid off, to scale, on the vertical immediately above the piston, as an ordinate from the axis X. Make $OD' = q$ by the same scale and draw the horizontal $D'C'$. Then the effective work done on the piston-rod while it moves thro' any small distance dx, is dW = force × space = $F(p-q)\,dx$, and is proportional to the area of the strip RS, whose width is dx and length $= p-q$; so that the effective work of one stroke is

$$\Big[_{0''}^{C''} W = \int_{x=0}^{x=l} dW = F\int_{x=0}^{x=l}(p-q)\,dx \ldots\ldots\ldots (1)$$

and is represented graphically by the area A'ARBCD'A'.

From O'' to D'' p is constant and $= p_b$, (while q is constant at all points), and x varies from 0 to a,

$$\therefore \Big[_{0''}^{D''} W = F(p_b - q)\int_0^a dx = F(p_b - q)\,a \ldots (2)$$

which may be called the work of entrance, and is represented by the ~~area of the~~ rectangle A'ADD'.

From D'' to C'' p is variable and, by Mariotte's law, $= \frac{a}{x}p_b$

$$\therefore \Big[_{D''}^{C''} W = F\int_a^l (p-q)\,dx = F\left[ap_b\int_a^l \frac{dx}{x} - q\int_a^l dx\right]$$

$$= F\left[ap_b \log_e\left(\frac{l}{a}\right) - q(l-a)\right]$$

the work of expansion, adding which to that of entrance, we have for the

TOTAL EFFECTIVE WORK OF ONE STROKE $\Big\} = W = Fp_b a\left[1+\log_e\left(\frac{l}{a}\right)\right] - Fql \ldots\ldots (4)$

If the engine is double-acting and makes n revolutions per time-unit, the work done on the piston-rod per unit of time, i.e. the POWER, is

$$(5) \ldots\ldots L = 2nW = 2nFap_b\left[1+\log_e\left(\frac{l}{a}\right)\right] - Fql$$

Example 1. A reciprocating steam engine makes 120 revolutions per minute, the boiler pressure is 40 lbs. by the gauge (i.e. $p_2 = 40 + 14.7 = 54.7$ lbs. per sq. inch), the piston area is $F = 1200$ sq. in., the length of stroke $l = 16$ in., the steam being cut off at $\frac{1}{4}$ stroke ($\therefore a = 4$ in., and $l/a = 4.00$) and the exhaust pressure corresponds to a "vacuum of 5 inches" (by which is meant that the pressure of the exhaust steam will balance 5 in. of mercury, whence $q = \frac{5}{30}$ of $14.7 = 2.45$ lbs. per sq. inch). Required the work per stroke, W, and the corresponding power L. Since $l : a = 4$, we have $\log_e 4 = 2.302 \times .60206 = 1.386$, and from eq. (4) (foot. lb. sec.)

$$W = \frac{1200}{144}(54.7 \times 144) \cdot \frac{1}{3} \cdot \left[2.386\right] - \frac{1200}{144}(2.45 \times 144) \frac{4}{3}$$

$= 5165.86 - 392.0 = 4773.868$ ft. lbs. of work per stroke

and $\therefore$ the power at 2 rev. per sec. $= n$ (eq. 5) is

$L = 2 \times 2 \times 4773.87 = 19095.5$ ft. lbs. per second.

and hence in horse-powers, which, in ft. lb. sec. syst., $= L \div 550$

$= 19095.5 \div 550 = 34.7$ H.P.

Example 2. Required the weight of steam consumed per second by the above engine with given data; assuming with Weisbach that the heaviness of saturated steam at a definite pressure (and a corresponding temperature, § 435) is about $\frac{5}{8}$ of that of air at the same pressure and temp.

The ~~heaviness~~ of air at 54.7 lbs. per sq. in. tension and temp 287° Fahr. (see table § 435) would be, from eq. (12) § 437, see also § 394 $\}$ ...

$$\gamma = \frac{\gamma_0 T_0}{T} \cdot \frac{p}{p_0} = \frac{.0807 \times 493}{461 + 287} \cdot \frac{54.7}{14.7}$$

$= 0.198$ lbs. per cub. foot, $\frac{5}{8}$ of which is 0.1237 lbs. cub. ft.

Now the volume of this heaviness admitted from the boil-

er at each stroke is $V' = Fa = \frac{120}{144}\cdot\frac{1}{3} = 0.2777$ cub. ft and hence the weight of steam used per second is $4 \times .2777 \times 0.1237 = 0.1374$ lbs.

Hence, per hour, $0.1374 \times 3600 = 494.6$ lbs. of feed-water are needed for the boiler.

[NOTE. For deviations from the above theory due to "wire-drawing", water mixed with the steam, etc., etc., see special works.]

444. GRAPHIC REPRESENTATION OF ANY CHANGE OF STATE OF A CONFINED MASS OF GAS. The curve of expansion AB in Fig. 518 is an equilateral hyperbola the axes X and Y being its asymptotes. If compressed air were used instead of steam its expansion curve would be the same if its temperature could be kept from falling during the expansion (by injecting hot water spray e.g.) and then, following Mariotte's law, we would have, as for the steam, (§ 440) pV = constant i.e. pFx = const, and ∴ px = constant, which is the eq. of a hyperbola, p being the ordinate and x the abscissa. This curve (dealing with a perfect gas) is also called an *isothermal*, the x and y co-ordinates of its points being proportional to the volume and tension, respectively, of a mass of air (or perfect gas) whose temp. is maintained constant.

Hence, in general, if a mass of gas be confined in a rigid cylinder of cr. section F (area), provided with an air-tight piston, Fig. 519, its volume, Fx, is proportional to the distance $OD = x$ (of the piston from the closed end of the cylinder) taken as an abscissa, while its tension p at the same instant may be laid off as an ordinate from D. Thus

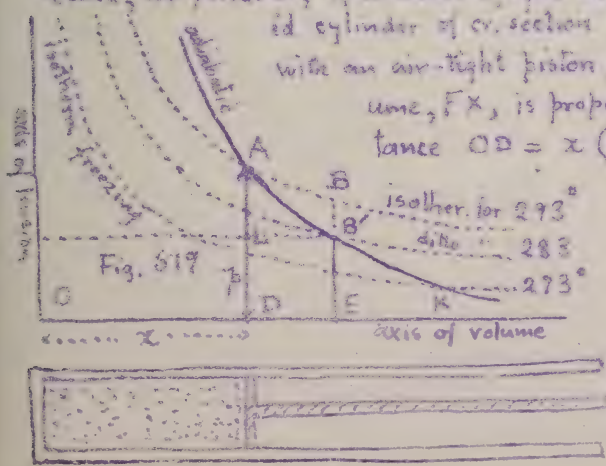


Fig. 519

point A is fixed. Describe an equilateral hyperbola thro' A, asymptotic to X and Y, and mark it with the observed temperature (absolute) of the air at this instant. In a similar way the diagram can be filled up with a great number of equilateral hyperbolas, or *isothermal curves*, each for its own temperature.

Any point whatever in the plane angular space YOX will indicate by its co-ordinates a volume and a tension, while the corresponding absolute temperature T will be shown by the hyperbola passing thro' the point, these three variables always satisfying the relation (§ 437) $\Big\}\ \dfrac{pV}{T} = \text{const. i.e. } \dfrac{pFx}{T} = \dfrac{p_0 V_0}{T_0} \ \ldots\ (1)$

Any change of state of the gas in the cylinder may now be represented by a line in the diagram connecting the two points corresponding to its initial and final states. Thus, a point moving along the line A B, a portion of the isothermal marked 293° Abs. Cent., represents a motion of the piston from D to E, and a consequent increase of volume, accompanied by just sufficient absorption of heat by the gas (from other bodies) to maintain its temperature at that figure (viz., its temp. at A). If the piston move from D to E, without transmission of heat, i.e. *adiabatically* (§ 442), the tension falls more rapidly, and a point moving along the line AB′ represents the corresponding continuous change of state. AB′ is a portion of an *adiabatic curve*, ~~whose~~ equation from § 442 is $\Big\}\ \dfrac{p}{p_0} = \left[\dfrac{Fx_0}{Fx}\right]^{\frac{3}{2}}$ or $p x^{3/2} = p_0 x_0^{3/2} = \text{const.} \ \ldots\ (1)$

in which p_0 and x_0 refer to the point K where this particular adiabatic curve cuts the isothermal of freezing point. Evidently an adiabatic may be passed thro' any point of the diagram. The mass of gas in the cylinder may change its state from A to B′ by an infinite number of routes, or lines of the diagram, the adiabatic route, however, being that most likely to occur for a rapid motion of the piston. For example,

we may cool it without allowing the piston to move, (and hence without altering its volume nor the abscissa x) until the pressure falls to a value $p_{B'} = DL = EB'$, and this change is represented by the vertical path from A to L; and then allow it to expand, and push the piston from D to E, (i.e. do external work) during which expansion heat is to be supplied at just such a rate as to keep the tension constant $= p_{B'} = p_L$, this latter change corresponding to the horizontal path LB', from L to B'.

It is further noticeable that the work done by the expanding gas upon the *near face* of the piston (or done upon the gas when compressed) when the space dx is described by the piston, being $= F p\,dx$, is proportional to the area $p\,dx$ of the small vertical strip lying between the axis X and the line or route showing the change of state; whence the total work done on the near piston face, being $= F \int p\,dx$, is represented by the area $\int p\,dx$, the plane figure between the initial and final ordinates, the axis X, and the particular route followed between the initial and final states (N.B. We take no account here of the pressure on the other side of the piston, the latter depending on the style of engine) For example, the work done on the near face of piston during adiabatic expansion from D to E is represented by the plane figure $AB'EDA$.

The mathematical relations between the quantities of heat imparted or rejected by conduction and radiation, or transformed into work, in the various changes of which the confined gas is capable, belong to the subject of *Thermodynamics* which can not be entered upon here.

It is now evident how the cycle of changes which a definite mass of air or gas experiences when used in a hot-air engine, compressed-air engine, or air compressor, is rendered more intelligible by the aid of such a diagram as Fig. 519; but it must be remembered that during the entrance into, or exit

from, the cylinder, of the mass of gas used in one stroke, the distance x does not represent its volume, and hence the locus of the points in the diagram determined by the co-ordinates p and x during entrance and exit does not indicate changes of state in the way just explained for the mass when confined in the cylinder. However, the work done by or upon the gas during entrance and exit will still be represented by the plane figure included by that locus (usually a straight horiz. line, press. constant) and the axis of X (and the terminal ordinates).

445. ADIABATIC EXPANSION IN AN ENGINE USING COMPRESSED AIR. Fig. 520.

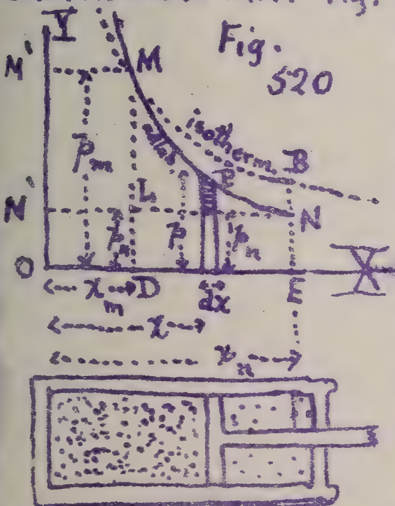


Fig. 520

Let the compressed air at a tension p_m and an absol. temp. T_m be supplied from a reservoir (in which the loss is continually made good by an air compressor). Neglecting the resistance of the port, its tension and temp. when behind the piston are still p_m and T_m. The cut-off is to be made at such a point D (of piston's progress), where $x = x_m$, and ratio $x_m : x_n$, x_n being whole length of stroke, that in adiabatic expansion from D to E the pressure shall have fallen from p_m at M (state m) to a value $p_n = p_a$ = one atmos. at N (state n), at the end of stroke, so that when the piston returns the air will be expelled ("exhausted") at a tension equal to that of the external atmos. (though at a low temp.). Hence the back-pressure at all points either way will be $= p_n$ per unit area of piston, and $\therefore$ total back-press. $= Fp_n$, F being the piston area.

From O to D the forward press. is constant and $= Fp_m$

$$\therefore \left.\begin{matrix}\text{WORK OF}\\ \text{ENTRANCE}\end{matrix}\right\} = \left[{}^{D}W = F\left[p_m - p_n\right] x_m \quad \ldots\ldots\ldots (1)\right.$$

represented by the rectangle $M'MLN'$. The cut-off being made at D, the volume of gas now in the cylinder, viz. $V_m = Fx_m$, is left to expand. Assuming no device adopted (such as injecting hot water spray) for preventing the cooling and rapid decrease of tension during expansion, the latter is adiabatic and the tension at any point P between M and N will be

$$p = p_m \left[\frac{x_m}{x}\right]^{\frac{3}{2}} \quad \cdots \left\{\begin{matrix}\text{see § 442} \\ V = Fx\end{matrix}\right\} \quad \cdots\cdots (1)$$

$$\therefore \left.\begin{matrix}\text{the WORK} \\ \text{OF EXPANSION}\end{matrix}\right\} = \left[{}_D^E W = \int_{x_m}^{x_n} F(p - p_n)\,dx = F\int_{x_m}^{x_n} p\,dx - Fp_n(x_n - x_m) \quad \cdots (2)$$

$$\text{But} \int_{x_m}^{x_n} p\,dx = p_m x_m^{\frac{3}{2}} \int_{x_m}^{x_n} x^{-\frac{3}{2}}\,dx = -2p_m x_m^{3/2}\left[\left(\frac{1}{x_n}\right)^{1/2} - \left(\frac{1}{x_m}\right)^{1/2}\right]$$

$$\text{i.e. } F\int_{x_m}^{x_n} p\,dx = 2Fp_m x_m\left[1 - \left(\frac{x_m}{x_n}\right)^{1/2}\right] \quad \cdots\cdots (3)$$

Now substitute (3) in (2), and add (1) and (2) note that

$$F(p_m - p_n)x_m - Fp_n(x_n - x_m) = Fp_m x_m\left[1 - \frac{x_n p_n}{x_m p_m}\right]$$

which, ~~since m and n are adiabatically related~~, $= Fp_m x_m\left[1 - \left(\frac{x_m}{x_n}\right)^{1/2}\right]$ and we have, finally,

$$\left.\begin{matrix}\text{total work per} \\ \text{stroke, on piston-rod}\end{matrix}\right\} = W = 3Fx_m p_m\left[1 - \left(\frac{x_m}{x_n}\right)^{1/2}\right] \quad \cdots (4)$$

But the adiab. relation $\left(\frac{V_m}{V_n}\right)^{1/2} = \left(\frac{p_n}{p_m}\right)^{\frac{1}{3}}$, i.e. $\left(\frac{x_m}{x_n}\right)^{1/2} = \left(\frac{p_n}{p_m}\right)^{1/3}$ enables us also, with $Fx_m = V_m$, to write

$$W = 3V_m p_m\left[1 - \left(\frac{p_n}{p_m}\right)^{1/3}\right] \quad \cdots (5)$$

in which V_m = the volume which the mass of air used per stroke occupies in the state m, i.e. in the reservoir where the pressure is p_m and the absolute temperature T_m. To find the work done per pound of

air used (or other unit of weight) we must divide W by the weight $G = V_m \gamma_m$ of the air used per stroke, remembering (eq. 13 § 437) that $V_m \gamma_m = \left[V_m p_m \gamma_0 T_0\right] \div (T_m p_0)$.

$\therefore$ work per unit of weight of air ~~used, in adiabatic expans.~~ working, when its tension falls

$$= 3 T_m \frac{p_0}{\gamma_0 T_0}\left[1 - \left(\frac{p_n}{p_m}\right)^{\frac{1}{3}}\right] \quad \ldots (6)$$

from p_m to p_n, p_n being = atmos. press. = the back-pressure.

In (6), γ_0 = .0807 lbs. per cub. ft., p_0 = 14.7 lbs. per sq. inch, and T_0 = 273° Abs. Cent. or 493° Abs. Fahr.

It is noticeable in (6) that for given tensions p_m and p_n, the work per unit of weight of air used is *proportional to the absolute temperature* T_m of the reservoir. The temperature T_n to which the air has cooled at the end of the stroke is obtained as in Example 2 § 442, and may be far below freezing point unless T_m is very high, or the ratio of expansion $x_m : x_n$ large.

Example. Let the cylinder of a compressed-air engine have a section of F = 108 sq. inches and a stroke of x_n = 15 inches. The compressed air entering the cylinder is at a tension of two atmospheres (i.e. p_m = 29.4 lbs. per sq. in., and $p_n \div p_m = \frac{1}{2}$) and a temperature of 27° Cent. (i.e. T_m = 300° Abs. Cent.) Required the proper point of cut-off, or x_m = ?, in order that the tension may fall to one atmos. at the end of the stroke; also the work per stroke, and the work per pound of air. Use the foot, lb., and sec., system of units.

From eq.(1) $\Big\}\; x_m = x_n\left(\frac{p_n}{p_m}\right)^{\frac{2}{3}} = 1.25\sqrt[3]{\frac{1}{4}} = 0.6875$ ft $= 8\frac{1}{4}$ inches

$\therefore$ Vol. of air in state m, used per stroke $\Big\}\; = V_m = F x_m = \frac{108}{144} \times .6875 = 0.515$ cubic ft.

While from eq.(5)

the work per stroke $\Big\}\; = W = 3 \times 0.515 \times 29.4 \times 144 \times \left[1 - \left(\frac{1}{2}\right)^{\frac{1}{3}}\right]$

= 1347.3 ft. lbs.; ~~while~~ and the work obtained from each lb. of air is, (eq. 5) $\Big\}\; = 3 \times 300 \times \frac{14.7 \times 144}{0.0807 \times 273}\left[1 - \sqrt[3]{\frac{1}{2}}\right] = 17810.0$ ft. lbs.

per lb. of air. The temperature to which the air has cooled at the end of stroke (eq. 2 § 442) is $T_n = T_m \sqrt[3]{\frac{p_n}{p_m}} = 300\sqrt[3]{\frac{1}{2}}$ $= 300 \times 0.794 = 238°$ Abs. Cent., that is, $-35°$ Centigrade.

446. REMARKS ON THE PRECEDING. This low temperature is objectionable, causing, as it does, the formation and gradual accumulation of snow, from the watery vapor usually found in small quantities in the air, and the ultimate blocking of the ports. By giving a large value to T_m, however, i.e., by heating the reservoir, T_n will be correspondingly higher, and also the work per pound of air, eq. (6). If the cylinder be encased in a jacket of hot water, or if spray of hot water be injected behind the piston during expansion, the temperature may be maintained nearly constant, in which event Mariotte's law will hold for the expansion, and more work will be obtained per pound of air, but the point of cut-off must be differently placed. Thus, if in eq. (4) § 443, we make the back-pressure equal to $(Fa \div Fl)\,p_b$ = the value to which the air-pressure has fallen at the end of stroke by Mariotte's law, we have

$$\left.\begin{array}{l}\text{Work per stroke}\\ \text{with isotherm. exp.}\end{array}\right\} = Fa\,p_b \log_\varepsilon\left(\frac{l}{a}\right) = V_b\,p_b \log_\varepsilon\left(\frac{l}{a}\right) \ldots (1)$$

$$\left.\begin{array}{l}\text{and } \therefore \text{ work per lb.}\\ \text{of air with isoth. exp.}\end{array}\right\} = T_m \frac{p_0}{\gamma_0 T_0} \log_\varepsilon\left(\frac{l}{a}\right) \ldots\ldots (2)$$

Applying these eq.s to the data of the preceding example, we obtain for isothermal expansion,

$$\left.\begin{array}{l}\text{Work per}\\ \text{lb. of air}\end{array}\right\} = 0.69\,T_m \frac{p_0}{\gamma_0 T_0}; \left.\begin{array}{l}\text{whereas with}\\ \text{adiab. exp.}\end{array}\right\} \text{it} \overset{\text{only}}{=} 0.62\,\frac{T_m\,p_0}{\gamma_0 T_0}$$

447. DOUBLE-ACTING AIR-COMPRESSOR, WITH ADIABATIC COMPRESSION. This is the converse of the foregoing article. In Fig. 521 we have the piston moving from right to left compressing the mass of air which at the

beginning of the stroke fills the cylinder. This is brought about by means of an external motor (steam engine or turbine, e.g.) which exerts a thrust or pull along the piston-rod, enabling it with the help of the atmos. press. of the fresh supply of air flowing in behind it, to first compress a cylinder-full of air to the tension of the compressed air in the reservoir, and then, the port or valve opening at this stage, to force or deliver it into the reservoir. Let the temperature and tension of the cylinder-full of fresh air be T_{n_1} and p_{n_1}, and the tension in the reservoir be p_{m_1}. Compression adiabatic.

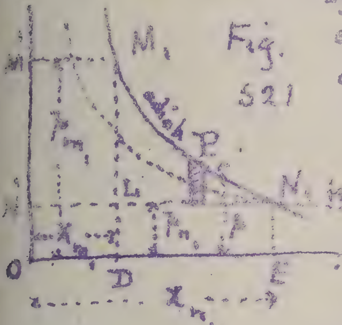
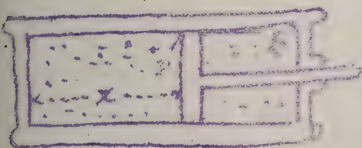


Fig. 521



As the piston passes from E toward the left, the air on the left has no escape and is compressed, its tension and temperature increasing adiabatically until it reaches a value p_{m_1} = that in reservoir, at which instant, the piston being at some point D, a valve opens and the further progress of the piston simply transfers the compressed air into the reservoir without further increasing its tension. Throughout the whole stroke the piston-rod has the help of one atmos. pressure on the right face, since a new supply of air is entering on the right, to be compressed in its turn on the return stroke. The work done from E to D may be called the work of compression, that from D to O, the work of delivery. Whence we have

$$\left.\begin{array}{l}\text{TOTAL WORK OF}\\ \text{ONE STROKE, BY PIS-}\\ \text{TON-ROD (i.e. extern. motor)}\end{array}\right\} = W = \underbrace{\int_E^D F(p - p_{n_1})\,dx}_{\text{comp.}} + \underbrace{F(p_{m_1} - p_{n_1})\int_D^O dx}_{\text{deliv.}} \quad \ldots (1)$$

In the adiabatic compress. the air passes from the state n_1 to the state m_1 (see N_1 and M_1 in figure).

The summation in eq. (1) being of the same form as that in eq.s (2) and (1) of § 445, with adiab. change from n_1 to m_1,

we may write immed.ly } ... $W = 3\,V_{m_1} p_{m_1}\left[1-\left(\frac{p_{n_1}}{p_{m_1}}\right)^{1/3}\right]$ .....(2)

and, WORK per unit of weight of air compressed } $= 3\,T_{m_1}\frac{p_0}{\gamma_0 T_0}\left[1-\left(\frac{p_{n_1}}{p_{m_1}}\right)^{1/3}\right]$ ...(3)

The value of T_{m_1}, at the (immed.) end of the sudden compression by eq. (2) of § 442 } is $T_{m_1} = T_{n_1}\left(\frac{p_{m_1}}{p_{n_1}}\right)^{1/3}$ ....(4)

The temperature of the reservoir being T_m as in § 445, (usually much less than T_{m_1}) the compressed air entering it cools down gradually to that temp., T_m, contracting in volume correspondingly since it remains at the same tension p_{m_1}. The mechanical equivalent of this heat is lost.

Let us now inquire what is the efficiency of the combination of air compressor and compressed-air engine, the former supply air for the latter, both working adiabatically, assuming that no tension is lost by the comp. air in passing along the reservoir between, i.e. that $p_{m_1} = p_m$. Also assume (as already implied, in fact) $p_{n_1} = p_n$ = one atmos., and that the temp. T_{n_1} of the air entering the compressor cylinder is equal to that T_m of the reservoir and transmission-pipe.

To do this we need only find the ratio of the amount of work obtained from one pound (or other unit of weight) in the compressed-air engine to the amount spent in compressing one pound of air in the compressor. Calling this ratio η, the efficiency, and dividing eq. 6 of § 445 by eq. 3 of this §, we have, with substitutions just mentioned } ... $\eta = \frac{T_m}{T_{m_1}} = \frac{\text{Abs. temp. of outer freezing}}{\text{Abs. temp. of air at end of sudden comp.}}$

or, substituting from eq. (4), and remembering that ....(5)

$T_{n_1} = T_m$ we have also $\eta = \left(\frac{p_n}{p_m}\right)^{1/3}$ ......(6)

also, since $\left(\frac{p_n}{p_m}\right)^{1/3} = \frac{T_n}{T_m}$ we may write $\Big\}$ $\therefore \eta = \frac{T_n}{T_m} = \frac{\text{Ab. Tem. air leav.}^g\text{ eng. cyl.}}{\text{Ab. Tem. outer free air}}$ (7)

Example 1. In the example of §445 the ratio of p_m to p_n was $= \frac{1}{2}$ Hence, if compressed air is supplied to the ~~reservoir under above~~ conditions, the efficiency of the system is from eq. (6) $\eta = \sqrt[3]{\frac{1}{2}} = 0.794$, about 80 percent.

Example 2. If the ratio of the Tensions is as small as $\frac{p_n}{p_m} = \frac{1}{6}$, the efficiency would be only $\left(\frac{1}{6}\right)^{1/3} = 0.55$; i.e. 45 per cent. of the energy spent in compressor is lost in heat

Example 3. What Horse Power is required in a blowing engine to furnish 10 lbs. of air per minute at a pressure of 4 atmos., with adiabatic compression, the air being received by the engin at one atmos. tension and 27° Cent. (Ft. lb. sec. system). Since 27° C = 300° Abs. C. = T_{n_1}, we have from equat. (4) $\Big\}$ $T_{m_1} = 300\left(\frac{4}{1}\right)^{1/3} = 477$ ° Abs. Cent.; and hence

eq. (3) the work per pound of air $\Big\}$ $= 3 \times 477 \frac{14.7 \times 144}{.0807 \times 273}\left[1 - \left(\frac{1}{4}\right)^{1/3}\right]$

$= 50870$ ft. lbs. per pound of air. Hence 10 lbs. of air will require 508700 ft. lbs. of work; and if this done every minute we have the req. H. P. $= \frac{508700}{33000} = 15.4$ H. P.

Note. If the compression could be made isothermal, an approximation to which is obtained by injecting a spray of cold water, we would have, from eq.s (1) and (2) of §446,

WORK per lb. air $\Big\}$ $= T_{n_1} \frac{p_0}{\gamma_0 T_0} \log_\varepsilon\left(\frac{p_{m_1}}{p_{n_1}}\right) = \frac{300 \times 14.7 \times 144}{.0807 \times 273} \times 1.386$

= 39952 ft. lbs. per lb., and the corresponding H.P. = 18.1
a saving of about 25 per cent., compared with the former.
The difference was employed in heating the air in the air-com-
pressor, with which comp. was lost when that extra heat was dissipated in
the reservoir as the air cooled again.

448. BUOYANT EFFORT OF THE ATMOSPHERE. In the case of large bulk but of small specific gravity the buoyant effort of the air (due to the same cause as that of water, see § 424) becomes quite appreciable and may sometimes be greater than the body's weight. This buoyant effort is equal to the weight of air displaced, i.e., $= V\gamma$, where V is the volume of air displaced and γ its heaviness.

If G_1 = total weight of the body producing the displacement the resultant vertical force is

$$P = G_1 - V\gamma \quad \text{.......} \quad (1)$$

and for equilibrium or suspension in the air, we must have

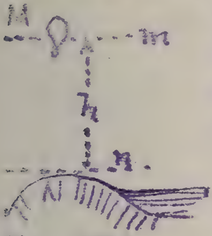
$$P = 0, \text{ i.e. } \ldots\ldots G_1 = V\gamma \quad \ldots\ldots \text{(equil.)} \ldots (2)$$

We may ∴ find approximately the elevation where a given balloon will cease to ascend, by determining the heaviness γ of the air at that elevation from eq. (2); then, knowing approximately the temperature of the air at that elevation we may compute its tension p (eq. 13 § 437) and finally from eqs (3) (4) or (5) of § 441, obtain the altitude required.

Example. The car and other solid parts of a balloon weigh 400 lbs., and the bag contains 12000 feet (cubic) of illuminating gas weighing 0.030 lbs. per cub. foot at a tension of one atmosphere, so that its total weight $= 12000 \times 0.030 = 360$ lbs. Hence $G_1 = 760$ lbs. We may also write with sufficient accuracy: whole volume of displacement $= V = 12{,}000$ cub. ft.

As the balloon ascends the exterior pressure diminishes, and the confined gas tends to expand and so increase the volume of displacement V; but this we shall suppose prevented by the strength of the envelope. At the surface of the ground

(station n of Fig. 522; see also Fig. 517) let the barometer read 29.6 in. and the temperature be 15° Cent. ($\therefore T_n =$ 288° Abs. Cent and the heav. of the air is



$$\gamma_n = \frac{.0807 \times 273}{14.7} \cdot \frac{\frac{29.6}{30}\,14.7}{288} = \frac{\gamma_0 T_0 p_n}{p_0 T_n} = .0807 \times \frac{273}{288} \cdot \frac{29.6}{30} = .0754$$

Fig. 522

lbs. per cub. ft. At the unknown height $= h$, where the balloon is to come to rest, i.e. at M, G_1 must $= V\gamma_m$, (eq. 2) or

$$\gamma_m = \frac{G_1}{V} = \frac{760 \text{ lbs.}}{12000 \text{ cub. ft}} = 0.0633 \text{ lbs. per cub. foot}$$

and if the temperature at M be estimated to be 5° Cent. (or $T_m =$ 278° Abs. Cent) (On a calm day the Temp. decreases about 1° Cent. for each 500 feet of ascent) we shall have, from

$$\frac{p_m}{\gamma_m T_m} = \frac{p_n}{\gamma_n T_n}, \quad \frac{p_n}{p_m} = \frac{\gamma_n}{\gamma_m} \cdot \frac{T_n}{T_m} = \frac{.0754}{.0633} \cdot \frac{288}{278} = 1.206$$

$\therefore$, eq. 5 § 441 (ft. lb. sec sys. is necessary) with the mean of T_m and T_n put for T_n

$$h = 26213 \, \frac{283}{273} \times 2.30258 \times \log_{10} 1.206 =$$

$= 5088.$ feet, the required height of ascent.

Chap. IV. Hydrodynamics begun; steady flow of liquids thro' pipes and orifices.

449. EXPERIMENTAL PHENOMENA OF A STEADY FLOW. As preliminary to the analysis on which the formulae of this chapter are based and to acquire familiarity with the tities involved, it will be advantageous to study the phenomena of the apparatus represented in Fig. 523. A large tank or reservoir BC is connected with another DE of a lower level, by means of a rigid pipe opening under the water-level in each tank. This pipe has no sharp curves

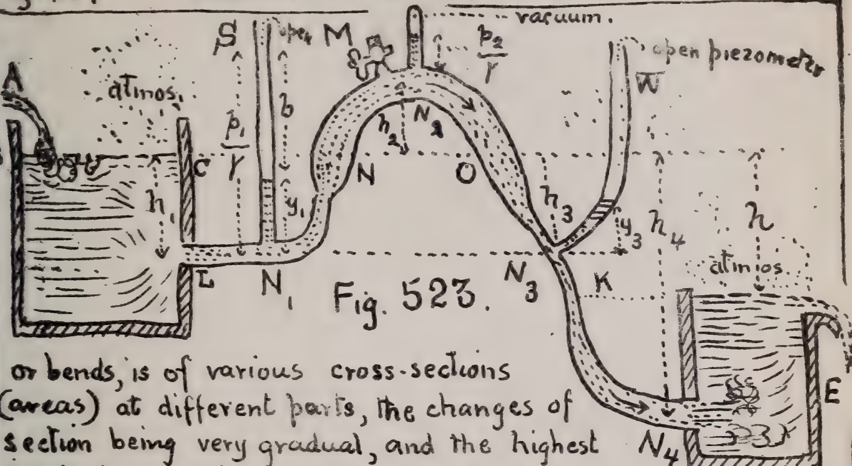


Fig. 523.

or bends, is of various cross-sections (areas) at different parts, the changes of section being very gradual, and the highest point N_2 not being more than 30 ft. higher than BC the surface level of the upper tank. Let both tanks be filled with water which will also rise to H and to K in the pipe. Stop the ends L and N_4 of the pipe, and thro' M, a stop-cock in the highest curve, pour in water to fill the remainder of the pipe; then, closing M, unstop L and N_4.

If the dimensions are not extreme (and subsequent formulae will furnish the means of testing such points) the water will now begin to flow from the upper tank into the lower, and all parts of the pipe will continue full of water as the flow goes on.

Further, suppose the upper tank so large that its surface-level sinks very slowly; or that an influx at A makes good, constantly, the efflux at E; then the flow is said to be a STEADY FLOW; or a *state of permanency* is said to exist; i.e., the circumstances of the flow at each section of the pipe are *permanent*, or *steady*.

By measuring the volume, V, of water discharged at E in a time t, we obtain the volume of flow per unit of time

viz. $$Q = \frac{V}{t} \quad \ldots\ldots\ldots\ldots (1)$$

while the weight of flow $\left\{\begin{matrix}\text{per unit} \\ \text{of time}\end{matrix}\right\} = G = Q\gamma \ldots (2)$

Water being incompressible and the pipe rigid, it follows that the same volume of water per unit of time must be passing at each cross-section of the pipe. But this is equal to the volume of a prism of water having F the area of the section as a base and, as an altitude, the mean velocity v with which the liquid particles pass through the section. Hence for all sections we have

$$Q = Fv = \text{a constant} = F_1 v_1 = F_2 v_2 = F_3 v_3 \text{, etc} \quad \left\{\begin{matrix}\text{EQ. OF}\\ \text{CONTIN-}\\ \text{UITY}\end{matrix}\right\} \quad (3)$$

in which the subscripts refer to different sections. If the flow were unsteady, e.g. if the level BC were sinking, this would be true for a definite instant of time; but when steady, we see that it is permanently true, e.g. that $F_1 v_1 = F_2 v_2$ at any instant subsequent or previous. In other words, *in a steady flow the velocity at a given section remains unchanged with lapse of time.* [N.B. We here assume for simplicity that the different particles of water passing simultaneously thro' a given section — abreast of each other have the same velocity one as another, (viz. the velocity which all other particles will assume on reaching this section). Strictly, however, the particles at the sides are somewhat retarded by friction on the surface of the pipe. This assumption is called the assumption of *Parallel Flow*, or *Flow in Plane Layers*, or *Laminated Flow.*]

Having, then, measured Q, we may, by knowing the area of the internal sections at the different parts of the pipe N_1, N_2, etc., compute the velocities $v_1 = Q \div F_1$, $v_2 = Q \div F_2$, etc., which the water must have in passing those sections, respectively. It is thus seen that the velocity at any section has no direct connection with the height or depth of the section from the plane BC, of the upper reservoir surface. The fraction $\frac{v^2}{2g}$ will be called the *height due to the velocity* v, or simply

the VELOCITY HEAD; for convenience.

Next, as to the value of the internal fluid pressure, p, per unit area, (in the water itself and against the inner surface of the pipe) at different sections of the pipe. If the end N_4 of the pipe were stopped, the problem would be one in hydrostatics, and the pressure against the side of the pipe at N_1 (also at N_3 at same level) would be simply $p_1 = p_a + h_1\gamma$, measured by a column of water of a height

$$\frac{p}{\gamma} = \frac{p_a}{\gamma} + h_1 = b + h_1,$$

{in which p_a = one atmos.; b = 34 feet = height of an ideal water barometer, and $\gamma \doteq 62.5$ lbs. per cub. foot.; and this would be shown experimentally by screwing into the side of the pipe at N_1 a small tube open at both ends; the water would rise in it to the level BC. That is, a column of water of height $= h_1$ would be sustained in it, which indicates that the internal pressure at N_1 corresponds to an ideal water column of height $= \frac{p_1}{\gamma} = b + h_1$. But when the steady flow is proceeding, the case being now one of hydrodynamics, we find the column of water sustained at rest in the small tube (called an open piezometer) N_1S has a height y_1, less than h_1, and hence the internal fluid pressure is less than it was when there was no flow. This pressure being p_1, the ideal water column measuring it is of a height .... $\frac{p_1}{\gamma} = b + y_1$ .... (4)

at N_1, and will be called the PRESSURE HEAD (at N_1 ~~pa~~ and similarly at any other section). We also find that while the flow is steady the piezometer height y_1 (and $\therefore$ also the pressure head, $= b + y_1$) remains unchanged with lapse of time.

At N_3, although at the same level as N_1, we find on inserting a piezometer W, that with $F_3 = F_1$ (and

∴ with $v_2 = v_1$) y_3 is a little less than y_1; while if $F_3 < F_1$ (so that $v_3 > v_1$) y_3 is not only less than y_1, but the difference is greater than before. We have ∴ found experimentally that in a general way when water is flowing in a pipe it presses less against the pipe, and the transverse laminae of the water exert less pressure against each other, than when at rest.

In the portion HN_2O of the pipe we find the pressure less than one atmosphere, and consequently a manometer registering pressures from zero upward, and not simply the excess above one atmos. (as with a steam-gauge, and also with the open piezometer just mentioned,) must be employed. At N_2 e.g. we find the press. $= \frac{1}{2}$ atmos. i.e. $\frac{p_2}{\gamma} = 17$ feet. Even below the level BC, by making the sections quite narrow (and consequently the velocities great) the pressure may be made less than one atmos. At the surface BC the pressure is of course just one atmos, while that in the jet at N_4 entering the small tank under water is necessarily $=$ one atmos. + press. due to column h', i.e.

$$\frac{p_4}{\gamma} = \text{pressure-head at } N_4 = b + h'$$

; (whereas if N_4 were stopped by a diaphragm, the pressure just on the right of the diaphragm would be $p_a + h'\gamma$, and on the left $p_a + h_4\gamma$.) Similarly, when a jet enters the atmosphere in parallel filaments its particles are under a pressure of one atmos. i.e., their pressure-head $=$ $b = 34$ feet; for the air immediately around the jet may be considered as a pipe between which and the water is exerted a press. of 14.7 lbs. per sq. inch.

450 RECAPITULATION AND EXAMPLES. We have found experimentally, then, that in a steady flow of liquid through a rigid pipe there is at each section of

the pipe a definite velocity and pressure which all the liquid particles assume on reaching that section; in other words at each section of the pipe the liquid velocity and pressure remain constant with lapse of time.

Example 1. If in Fig. 523, the flow having become steady, the volume of water flowing in 3 minutes is found by measurement to be 134 cubic feet, the volume per sec. and is, eq.(1) § 449, $Q = \frac{134}{180} = 0.744$ cub. ft. per sec.

Example 2. If the flow in 2 min 20 sec. is 386.4 the volume of flow per sec. is [ft. lb. sec. eqs. (1) and (2)]

$$Q = \frac{V}{t} = \frac{G}{\gamma} \div t = \frac{386.4}{62.5} \cdot \frac{1}{140} = 0.0441 \text{ cub. ft. per sec.}$$

Example 3. In Fig. 523 the height of the open piezometer at N_1 is $y_1 = 9$ feet; what is the internal fluid pressure at N_1? (inch-lb. sec.) The internal pressure is

$$p_1 = p_a + y_1\gamma = 14.7 + 108 \frac{62.5}{1728} = 18.6 \text{ lbs per sq. in.}$$

The pressure on the outside of the pipe is of course one atmosphere, so that the resultant bursting pressure at that point (N_1) is 3.9 lbs. per sq. inch.

Example 4. The volume of flow per sec. being .0441 cub. ft, as in Ex. 2, required the velocity at a section where the diameter is two inches (ft. lb. sec).

$$v = \frac{Q}{F} = \frac{0.0441}{\frac{1}{4}\pi\left(\frac{2}{12}\right)^2} = 2.02 \text{ ft. per second};$$

while at another section of the pipe where the diam. = 4 inches and F is four times as great, the veloc. = $\frac{1}{4}$ of 2.02 = 0.505 ft. per sec.

451. BERNOULLI'S THEOREM FOR STEADY FLOW; without friction.

If the pipe is comparatively short, without sudden bends, elbows, or abrupt changes of cross-section, the effect of friction of the liquid particles against the sides of the pipe, and against each other (…

eddies are produced, disturbing the parallelism of the ... small and will be neglected in the present ... chief object is to establish a formula for steady flow thro' a short pipe and thro' orifices.

An assumption now to be made of flow in ~~plane~~ layers or laminae $\perp$ to the pipe's axis at every point, may be thus stated; see Fig. 523a: All the liquid particles which at any instant form a small lamina, as AB, $\perp$ to pipe's axis, keep company as a lamina throughout the whole flow. The thickness ds' of this lamina remains constant so long as the pipe is of constant cross-section, but shortens up (e.g. at C) on passing thro' a larger section, and lengthens out (e.g. at D) in a part of the pipe where the sectional area, F, is smaller. The mass of such a lamina $= F\,ds'\gamma \div g$ (§ 55), its velocity at any section is v (pertaining to that point of the pipe's axis), the pressure of the lamina just behind it is Fp, upon its rear face, while the resistance (at same instant) met from its neighbor just ahead is $F(p+dp)$ on its front face; also its weight is the vertical force $F\,ds'\gamma$.

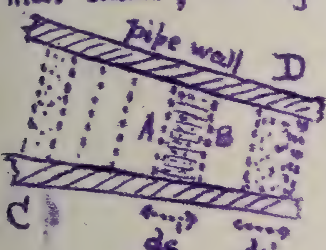


Fig. 523 a

Fig. 524 shows as a free body the lamina which at any instant is passing a point A, of the pipe's axis, where the velocity is v, and pressure p. Note well the forces acting; the pressures of the pipe on the edges of the lamina have no components in direction of v, i.e. of the tangent. (no fric. considered.) Apply to this free body eq. 7 of § 74, for any instant of any

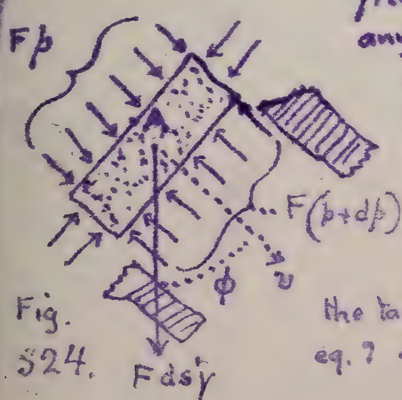


Fig. 524.

curvilinear motion of a material point, viz.: $\Big\}\ vdv =$ tang. acceleration $\times ds$ ...(1)

in which ds = a small portion of the path and is described in the time dt. Now the tang. accel = tang. components of the forces acting ÷ mass of lamina, i.e.

$$\text{tang. accel.} = \frac{Fp - F(p+dp) + F\gamma ds' \cos\phi}{F\gamma ds' \div g} \quad \ldots(2)$$

Now, Fig 525, at a definite instant of time, conceive the volume of water in the pipe to be subdivided into a great number of laminae of equal mass (which implies equal volume in the case of a liquid, but not with gaseous fluids) and let the ds just mentioned for any one lamina be the distance from its

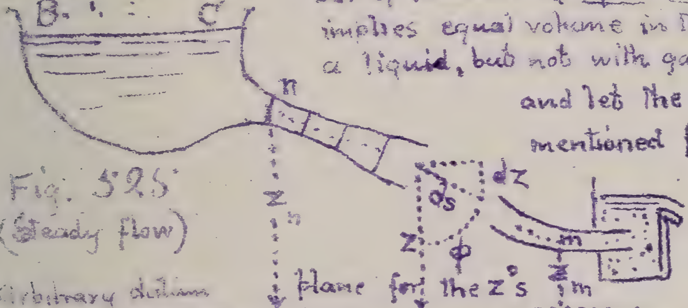


Fig. 525
(Steady flow)

centre to that of the one next ahead; this mode of subdivision makes the ds of any one lamina identical in value with its thickness ds', i.e., $ds = ds'$ -------(3)

We have, also, $ds \cos\phi = ds' \cos\phi = -dz$ ...(4)

z being the height of the centre of a lamina above any convenient "datum plane", horizontal. Substituting from (2), (3), (4) in eq. (1) we finally derive, after transposition,

$$\frac{1}{g}\, vdv + \frac{1}{\gamma}\, dp + dz = 0 \quad \ldots\ldots(5)$$

The flow being steady, and as a consequence of the mode of subdivision just made, each lamina in the time dt moves into the position which at the beginning of dt was filled by the lamina next ahead and acquires the same velocity, the same pressures on its faces, and the same value of z, that the front lamina had at the beginning of

dt. Hence, considering the simultaneous advance made by all the laminae in this one dt, we may write out an equation like (5) for each of the laminae between any two cross sections n and m of the pipe, thus obtaining an infinite number of equations, from which by adding corresponding terms, i.e. by integration, we obtain

$$\frac{1}{g}\int_{v_n}^{v_m} v\,dv + \frac{1}{\gamma}\int_{p_n}^{p_m} dp + \int_{z_n}^{z_m} dz = 0 \; ; \ldots (6)$$

whence, performing the integrations, and transposing,

$$\frac{v_m^2}{2g} + \frac{p_m}{\gamma} + z_m = \frac{v_n^2}{2g} + \frac{p_n}{\gamma} + z_n \ldots \left\{ \begin{array}{l} \text{BERNOUL-} \\ \text{LI'S THEOREM} \end{array} \right\} (7)$$

Denoting by *Potential Head* the vertical height of any section of the pipe above a convenient datum level, we state Bernoulli's Theorem as follows:

In steady flow without friction, the sum of the velocity head, pressure-head, and potential head at any section of the pipe is a constant quantity, being equal to the sum of the corresponding heads at any other section.

It is noticeable that in eq. (7) each of the terms is a linear quantity, viz. a height, or head, either actual, such as z_n and z_m, or ideal (all the others) and does not bring into account the absolute size of the pipe nor even its relative dimensions (except that we know from the eq. of continuity that $F_m v_m = F_n v_n$) and contains no reference to the volume of water flowing per unit of time [Q] or the shape of the pipe's axis. When the pipe is of considerable length compared with its diameter the friction of the water on the sides of the pipe cannot be neglected (§ 470).

It must be remembered that Bernoulli's Theorem does not hold unless the flow is steady, i.e., unless each lamina into the position just vacated by the one next ahead

(of equal mass) comes also into the exact conditions of velocity and pressure in which the other was when in that position.

452. FIRST APPLICATION of *Bernoulli's Theorem without friction.* Fig. 526 shows a large tank from which a vertical pipe of uniform section leads to another tank, dipping below the water-surface in the latter. Both water surfaces are open to the air. The vessel and pipe being filled with water, and the lower end m unstopped, a steady flow is established almost immediately, the surface BC being very large compared with F the area of the section of the pipe. Given F, and the heights h_0 and h, required the velocity v_m of the jet at m and also the pressure at n, viz. p_n. m is in the jet just clear of the pipe, and practically in the water level AD. The velocity at $m = v_m$ is unknown, but the pressure p_m is practically $= p_a =$ one atmos, since the pressure on the sides of the jet is necessarily the hydrostatic pressure belonging to a slight depth below the surface AD;

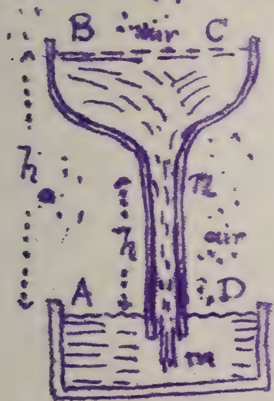


Fig. 526

$\therefore \dfrac{p_m}{\gamma} = \dfrac{p_a}{\gamma} = b = 34$ feet, is the pressure-head at m (§ 407). Now applying Bernoulli's theorem to sections m and n, taking a horiz. plane thro' m as a datum plane for potential heads, so that $z_n = h$ and $z_m = 0$, we have

$$\frac{v_m^2}{2g} + b + 0 = \frac{v_n^2}{2g} + \frac{p_n}{\gamma} + h \quad \text{......} \quad (1)$$

But, assuming that the section of the pipe is filled at every point, $\therefore$, since $F_m = F_n = F$, v_m must $= v_n$ (see eq. of continuity $F_n v_n = F_m v_m$) and hence (1) reduces to

$$\frac{p_n}{\gamma} = b - h = 34\text{ ft.} - h \quad \ldots\ldots \quad (2)$$

Hence the pressure at n is less than one atmosphere, and if a small tube communicating with an airtight receiver full of air were screwed into a small hole at n, the air in the receiver would gradually be drawn off until its tension had fallen to a value p_n. [This is the principle of Sprengel's air-pump, mercury, however, being used instead of water, as for this heavy liquid b = only 30 inches.]

If h is made $> b$ for water i.e. > 34 feet (or $>$ 30 inches for mercury) p_n would be negative from eq. (2) which is impossible, showing that the assumption of full pipe-sections is not borne out. In this case, $h > b$, only a portion mn' of the tube will be filled with water Fig. 526a. (somewhat $< b$); and in the part Kn' vapor of water, of low tension corresponding to the temperature, (§435) will surround an internal jet which does fill the pipe.

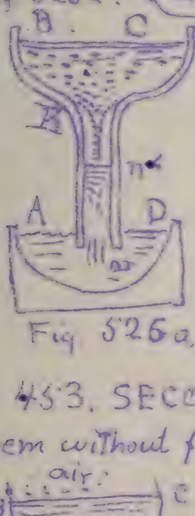


Fig. 526a.

Example. If h = 20 feet, Fig. 525, and the liquid is water the pressure-head at n is (ft. lb. sec.)

$$\frac{p_n}{\gamma} = b - h = 34 - 20 = 14^{ft} \text{ and }$$

$p_n = 14 \times 62.5 = 875$ lbs. per sq. ft = 6.07 lbs. per sq. in.

453. SECOND APPLICATION OF *Bernoulli's Theorem without friction.* Knowing by actual measurement the open piezometer height y_n at the section n in Fig. 527 (so that the pressure-head $\frac{p_n}{\gamma} = b + y_n$ at n is known;

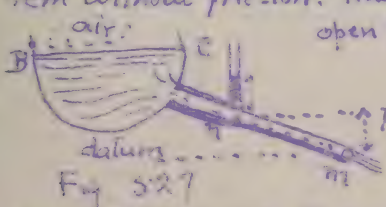


Fig. 527

Knowing also the vertical distance h_n from n to m, the respective cross sections F_n and F_m, (F_m being the sectional area of the jet, flowing

into the air so that $\frac{p_m}{\gamma} = b$); required the volume of flow per second, viz. $Q = F_n v_n = F_m v_m$ ..(1)

The pipe is short, with smooth curves if any, and friction will ∴ be neglected. From Bernoulli's Theorem (eq. (7) § 451) taking m as a datum plane for potential heads we have

$$\frac{v_m^2}{2g} + b + 0 = \frac{v_n^2}{2g} + (y_n + b) + h_n \quad \cdots \quad (2)$$

But from (1) we have $v_n = (F_m \div F_n) v_m$: substituting which in (2) gives, solving for v_m } .. $v_m = \dfrac{\sqrt{2g(y_n + h_n)}}{\sqrt{1 - \left(\frac{F_m}{F_n}\right)^2}}$ .... (3) and hence the volume per second becomes known, viz.: } $Q = F_m v_m$ .... (4)

[*Note* If the cross-section F_m of the nozzle, or jet, is $> F_n$, v_m becomes imaginary; unless y_n is negative (i.e. $p_n <$ one atmos.) and is numerically greater than h_n; in other words the assigned cross-sections are not filled by the flow.]

Example. If $y_n = 17$ feet (thus showing the internal fluid pressure at n to be $p_n = \gamma(y_n + b) = 1\frac{1}{2}$ atmos.), $h_n = 10$ feet, and the (round) pipe is 4 inches in diameter at n, and 3 in. at the nozzle m, we have from (3) (using ft. lb. sec. system in which $g = 32.2$)

$$v_m = \frac{\sqrt{2 \times 32.2(17 + 10)}}{\sqrt{1 - \left[\frac{\frac{1}{4}\pi\, 3^2}{\frac{1}{4}\pi\, 4^2}\right]^2}} = 50.4 \text{ ft. per sec.}$$

and ∴ the volume per sec. is $Q = F_m v_m =$

$= \frac{1}{4}\pi\left(\frac{3}{12}\right)^2 \times 50.4 = 2.474$ cub. ft. per second.

454. ORIFICES IN A THIN PLATE. Fig. ~~525~~ 527

When efflux takes place through an "orifice in a thin

plate", i.e., a sharp edged orifice in the plane wall of a tank, a *contracted vein* (or *vena contracta*) is formed, the filaments of water not becoming parallel until reaching a plane, m, which for circular orifices is at a distance from the interior plane of the wall equal to the radius of the circular aperture, and not until reaching this plane does the internal fluid pressure become equal to that of the medium (atmos. here) surrounding the jet). The cross section of the jet at m, called the *contracted section*, is found on measurement to be from 0.60 to 0.64 of the aperture, in area, with most orifices of ordinary shapes, even with a considerable range in the size of the orifice and in the height or *head*, h, producing the flow. Calling this abstract number the CO-EFFICIENT OF CONTRACTION, and denoting it by C, we may write

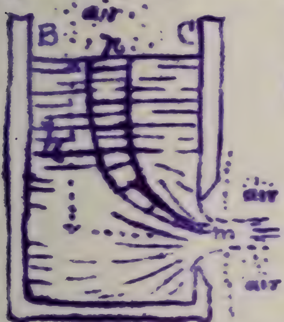


Fig. 527

$$F_m = CF$$

in which F = area of the orifice, F_m = that of the contracted section, and C = from .60 to .64 in most practical cases.

A lamina of particles of water is under atmos. pressure at n (the free surface of the water in the tank) while its velocity at n is practically v_n = zero (the surface BC being very large compared with the orifice). It experiences increasing pressure as it slowly descends, until in the immediate neighborhood of the orifice, when its velocity is rapidly accelerated and pressure decreased, in accordance with Bernoulli's theorem, and its shape lengthened out, until finally at m it forms a portion of a filament of the jet, its pressure is one atmosphere, and

its velocity $= v_m$ we wish to determine. The course of this lamina we call a "stream-line", and Bernoulli's theorem is applicable to it, just as if it were enclosed in a frictionless pipe of the same form. Taking then a datum plane through the centre of m, we have $\frac{p_m}{\gamma} = b$, $z_m = 0$, and $v_m = ?$; while $\frac{p_n}{\gamma}$ also $= b$, $z_n = h$ and $v_n = 0$. Hence Bern. Th. gives

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h, \therefore \frac{v_m^2}{2g} = h \quad \ldots\ldots(1)$$

and $v_m = \sqrt{2gh}$

That is, the velocity of the jet at m is theoretically the same as that acquired by a body falling freely in vacuo through a height $= h =$ "the head of water." We should $\therefore$ expect that if the jet were directed vertically upward, as at m Fig. 528, a height $= \frac{v_m^2}{2g}$ would be actually attained (§§ 52 and 53). Experiment shows that the height of the jet does not appreciably differ from h if h is not $>$ 6 or 8 ft.

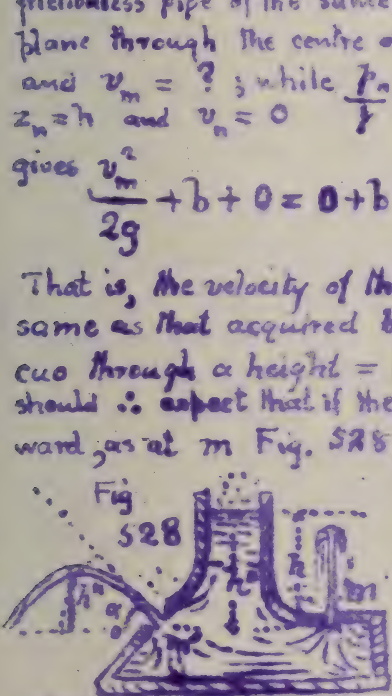


Fig 528

For $h > 8$ ft., however, the actual height reached is less than h, (not only absolutely, but relatively, greater as the difference being h increases, since the resistance of the air is more and more effective in depressing and breaking up the stream.

At m', Fig. 528, we have a jet directed at an angle α_0 with the horizontal. Its form is a parabola (§81) and the theoretical height, h'', reached is $h'' = h' \sin^2 \alpha_0$ (§80).

The jet from an orifice in thin plate is very limpid and clear.

From eq.(1) we have theoretically $v_m = \sqrt{2gh}$ (as we shall always write for efflux into the air through orifices and short pipes in the plane wall of a large tank, whose water sur-

face is very large compared with the orifice and is open to the air) but experiment shows that for an orifice in a thin plate this value is reduced about 3 % by friction at the edges, so that for ordinary practical purposes we may write

$$v_m = \phi\sqrt{2gh} = 0.97\sqrt{2gh} \quad \ldots \ldots (2)$$

in which ϕ is called the coefficient of velocity. $\therefore$ the volume of flow per unit of time Q, will be

$$Q = F_m v_m = CF\phi\sqrt{2gh} = \left\{\begin{matrix}\text{on the}\\ \text{average}\end{matrix}\right\} 0.62\, F\sqrt{2gh} \ldots (3)$$

It is understood that the flow is steady, and that the reservoir surface (very large) and the jet are both under atmospheric pressure. ϕC = co-efficient of efflux.

Example 1. Fig. 527. Required the velocity of efflux v_m, at m, and the volume of flow per second, Q, into the air, if $h = 21$ ft 6 in, the circular orifice being 2 in. in diameter; take $C = 0.64$. Ft. lb. sec.

From eq. (2) $v_m = 0.97\sqrt{2 \times 32.2 \times 21.5} = 36.1$ ft per sec.

$\therefore$ the discharge is

$$Q = F_m v_m = 0.64 \times \frac{\pi}{4}\left(\frac{2}{12}\right)^2 \times 36.1 = 0.504 \text{ cub. ft. per sec}$$

Example 2 (Weisbach). Under a head of 3.396 metres the velocity v_m in the contracted section is found by measurements of the jet-curve to be 7.98 metres per sec., and the discharge proves to be 0.01825 cub. metres per sec. Required the co-efficients of velocity and of contraction, ϕ and C, if the area of the orifice is 36.3 sq. centimetres. Use the met., kilo., sec. system of units, in which $g = 9.81$ met. per sq. sec.

From eq. (2), $\dfrac{7.98}{\sqrt{2 \times 9.81 \times 3.396}} = \phi = \dfrac{v_m}{\sqrt{2gh}} = 0.978$ (abstract numb.) while from (3) we have

$$C = \frac{Q}{F\phi\sqrt{2gh}} = \frac{Q}{Fv_m} = \frac{.01825}{\frac{363}{10000}\times 7.98} = 0.631$$

ϕ and C, being abstract numbers, are independent of the system of concrete units used.

455. ORIFICE WITH ROUNDED APPROACH.

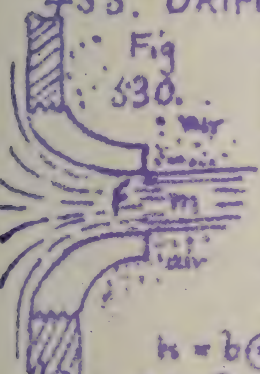


Fig. 530.

Fig. 530 shows the general form and proportions of an orifice or mouth-piece in the use of which contraction does not take place beyond its edges, the inner surface being one of "revolution" and so shaped that the liquid filaments are parallel on passing the outer edge m; hence the pressure-head at m is = b (= 34 ft. for water and 30 in. for mercury) in Bernoulli's Theorem, if efflux takes place into the air. Also the section F_m is equal to F that of the orifice, i.e. the co-efficient of contraction is C = 1.0 so that the discharge per unit of time has a volume $Q = F_m v_m = F v_m$. The tank being large etc., as in Fig. 528, Bernoulli's theorem applied to m and n, gives, as before, $v_m = \sqrt{2gh}$ as a theoretical result, while practically

$$v_m = \phi\sqrt{2gh} \quad \ldots(1) \quad \text{and} \quad Q = F\phi\sqrt{2gh} \quad \ldots(2)$$

As an average ϕ is found to differ little from 0.97 with this orifice, the same value as with the orifice in a thin plate.

456. PROBLEMS OF EFFLUX, SOLVED BY APPLYING BERNOULLI'S THEOREM. In the two preceding §§ the pressure-heads at sections m and n were each = $\frac{p_a}{\gamma}$ = 34 feet for water; but in the following problems this will not be the case necessarily. However, the efflux is to take place through a simple orifice in the side of a large reservoir, whose

upper surface (n) is very large so that v_n may be put = zero.

Problem I. Fig. 530. What is the velocity of efflux, v_m, at the orifice m (i.e. at the contracted section, if it is an orifice in thin plate) of the water from the side of a steam boiler, if the free surface at n is a height h above m, and the pressure of the steam over the water is p_n, the jet discharging into the air?

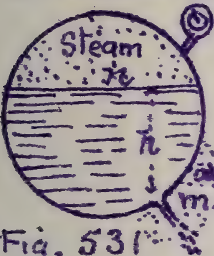


Fig. 531

Applying Bernoulli's theorem to section m at the orifice, (where the pressure-head is b, and velocity-head $v_m^2 \div 2g$ (unknown)), and to section n at water surface (where veloc. head = zero and the pressure head $= p_n \div \gamma$), we have, taking m as a datum for potential heads so that $z_m = 0$, $z_n = h$;

$$\frac{v_m^2}{2g} + b + 0 = 0 + \frac{p_n}{\gamma} + h; \quad \therefore v_m = \sqrt{2g\left[\frac{p_n}{\gamma} - b + h\right]} \quad \ldots (1)$$

Example. Let the steam-gauge read 40 lbs. (and $\therefore p_n =$ 54.7 lbs. per sq. inch) and $h = 2^{ft}\,4^{''}$; required v_m. Also if $F =$ 2 sq. inch, in thin plate, $Q =$? For ~~variety~~ use the inch-lb. sec. system of units. in which $g =$ 386.4 ~~inches per second~~, $b = 408$ inches, and the ~~heaviness~~ $\gamma = [62.5 \div 1728]$ lbs. per cubic inch.

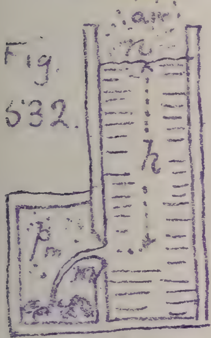
From (1) $\Big\}$ $v_m = \sqrt{2 \times 386.4 \left[\dfrac{54.7}{62.5 \div 1728} - 408 + 28\right]}$

$= 935.3$ inches per second, theoretically; practically $v_m = 0.97 \times 935.3 = 907.$ inches per sec. and the volume discharged per sec. $= Q = 0.64 F v_m = 0.64 \times 2 \times 907.$

= 1160, 96 cubic inches per sec.

PROBLEM II. Fig. 532. With what velocity, v_m, will water flow into the condenser of a steam engine, where the tension of the vapor is p_m < one atmos., if h = head of water and the flow takes place thro' an orifice in thin plate. Taking position m in the contracted section where the filaments are parallel and the pressure = p_m = that of the surrounding vapor; and position n in the (wide) free surface of the tank, where the pressure is one atmos. (and ∴ $\frac{p_n}{\gamma} = b = 34$ ft.) and veloc. practically zero, we have, applying Bernoulli's theorem to n and m, take m as a datum level for potential heads so that $z_n = h$, $z_m = 0$,

Fig. 532.



$$\frac{v_m^2}{2g} + \frac{p_m}{\gamma} + 0 = 0 + b + h \; ; \therefore v_m = \sqrt{2g\left[h + b - \frac{p_m}{\gamma}\right]} \quad \text{.... eq. (1)}$$

and $Q = F_m v_m$ ...(2) as theoretical results.

Practically we have $v_m = 0.97\sqrt{2g\left[h+b-\frac{p_m}{\gamma}\right]}$ ...(3)

and $Q = F_m v_m$ .... (4), in which $F_m = F$ = area of orifice, if rounded, but = CF if in thin plate, C being the co-ef. of contract., = about 0.62

Example. If in the condenser there is a "vacuum of $2\frac{1}{2}$ inches" (meaning that the tension of the vapor would support $2\frac{1}{2}$ in. of mercury; so that $p_m = \left[\frac{2.5}{30} \times 14.7\right]$ lbs. per sq. inch) and $h = 12$ feet, while the orifice has a diameter of $\frac{1}{2}$ inch; with the ft., lb., and sec., we shall have

$$v_m = 0.97\sqrt{2 \times 32.2\left[12 + 34 - \frac{\frac{5}{60}\,14.7 \times 144}{62.5}\right]}$$

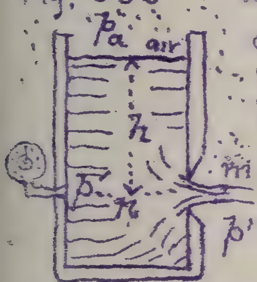
= 51.1 ft. per sec. (We might also have written,

for brevity, $\frac{p_m}{\gamma} = \left[2\frac{1}{2} : 30\right] \times 34 = 2.833$ feet) the pressure head for one atmosphere = 34 feet, for water.) Hence, circul. orifice in thin plate,

$$Q = CFv_m = 0.62 \times \frac{\pi}{4}\left(\frac{1/2}{12}\right)^2 \times 51.1 = .0431 \text{ cub. ft. per sec.}$$

457. EFFLUX THRO' AN ORIFICE IN TERMS OF INTERNAL AND EXTERNAL PRESSURES. Let efflux take place thro' a small orifice from the plane side of a large tank, in which at the level of the orifice the hydrostatic pressure was $= p'$ before the opening of the orifice, while that of the medium surrounding the jet is $= p''$.

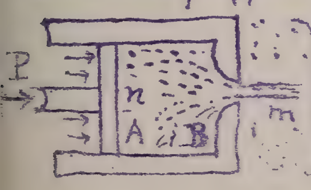
Fig. 533



When the steady flow is established, after opening the orifice, the pressure p' in the water on a level with the orifice, will not be materially changed except in the immediate neighborhood of the orifice (see § 454); hence applying Bernoulli's theorem (§ 451) to m in the jet where the filaments are parallel, and a point n in the body of the liquid at the same level as m and where the particles are practically at rest (i.e. $v_n = 0$), hence not too near the orifice, we shall have (the potential heads, being equal, cancel out)

$$\frac{v_m^2}{2g} + \frac{p''}{\gamma} = \frac{0^2}{2g} + \frac{p'}{\gamma} \quad \therefore v_m = (0.97)\sqrt{2g\left[\frac{p' - p''}{\gamma}\right]} \quad \text{eq. (1)}$$

(In Fig. 533 p' would $= p_a + h\gamma$) Eq. (1) is conveniently applied to the jet produced by a FORCE-PUMP, when the orifice is in the head of the pump cylinder, Fig. 534. Let the force P be applied along the piston rod, and the area of the piston be $= F'$. Then



the intensity of internal pressure produced in the chamber AB, when the piston moves uniformly, is $p' = \frac{P + F p_a}{F'}$ while the external pressure in the air around the jet is p_a (one atmos.) $\therefore\ v_m = 0.97\sqrt{2g\frac{P}{F'\gamma}}$ ...(1)'

Example. Let the force, or thrust, P, due to a steam pressure in a cylinder not shown in the figure, be 2000 lbs., and the diameter of pump-cylinder be $d =$ 9 inches, the liquid being salt water ($\gamma = 64$ lbs. per cubic foot. $\therefore F' = \frac{1}{4}\pi\left(\frac{9}{12}\right)^2 = 0.442$ sq. ft. and (ft. lb. sec.)

$$v_m = 0.97\sqrt{2 \times 32.2 \times \frac{2000}{0.442 \times 64}} = 65.4 \text{ ft. per sec.}$$

If the orifice is well rounded, with a [illegible] of one inch, the volume discharged per second is

$$Q = F_m v_m = F v_m = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 \times 65.4 = .353 \text{ cub. ft. per sec.}$$

To maintain this discharge steadily, the piston must move at the rate of $v' = \frac{Q}{F'} = .353 \div \frac{\pi}{4}\left(\frac{9}{12}\right)^2 = 0.8\cancel{0}6$ ft. per sec.

and the force P must exert a power of

$$L = 2000 \times 0.816 = 1632 \text{ ft. lbs. per second}$$
$$= \text{about } 3 \text{ H.P. (Horse power)}$$

If the water must be forced from the cylinder through a pipe or hose before passing out of a nozzle into the air, the velocity of efflux will be smaller, on account of friction in the hose, for the same P. Such a problem will be treated in a subsequent paragraph (§ 471)

458. INFLUENCE OF DENSITY ON THE VELOCITY OF EFFLUX IN THE LAST PROBLEM. From the equation $v_m = \sqrt{2g\frac{p' - p''}{\gamma}}$ of the preceding

§, where p'' is the external pressure around the jet, and p' the internal pressure at the same level as the orifice but well back of it where the liquid is sensibly at rest, we notice that for the same difference of pressure $(p'-p'')$ the velocity of efflux is inversely proportional to the square root of the heaviness of the liquid. Hence for the same $(p'-p'')$ mercury would flow out of the orifice with a velocity $\sqrt{\frac{62.5}{848}} = \sqrt{\frac{1}{13.5}} = \frac{272}{1000}$ of that of water; while, assuming that the equation held good for the flow of gases, as it does approximately when p' does not greatly exceed p'' (e.g. by 6 or 8 %), the velocity of efflux of air when at a heaviness of .0807 lbs. per cubic foot would be $\sqrt{\frac{62.5}{.0807}} = \sqrt{775.3} = 27.8$ times as great as for water. See § 501.

459. EFFLUX UNDER WATER, SIMPLE ORIFICE Fig. 535. Let h_1 and h_2 be the depths of the (small) orifice below the levels of the "head" and "tail" waters respectively. Then, using the formula of § 457, we have the pressure at n (at same level as m the jet) $p' = (h_1 + b)\gamma$ and the external pressure (around the jet at m) $p'' = (h_2 + b)\gamma$, whence theoretically

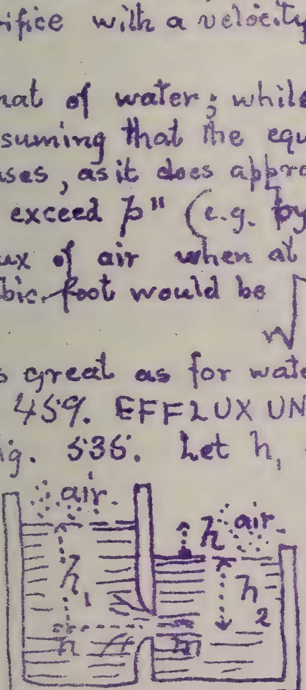


Fig. 535

$$v_m = \sqrt{2g\,\frac{p'-p''}{\gamma}} = \sqrt{2g\,h_1 - h_2} = \sqrt{2gh} \quad ..(1)$$

where h = difference of level of the two bodies of water. Practically $v_m = \phi\sqrt{2gh}$ ------ (2)

but the value of ϕ for efflux under water is some-

what uncertain; as also that of C the co-efficient of contraction. Weisbach says that $\mu_3 = \phi C_3$ is $1/75$ part less than for efflux into the air; others that there is no difference (Trautwine)

460. EFFLUX FROM A SMALL ORIFICE IN A VESSEL IN MOTION. *Case I. When the motion is a vertical translation and uniformly accelerated.* Fig. 536.

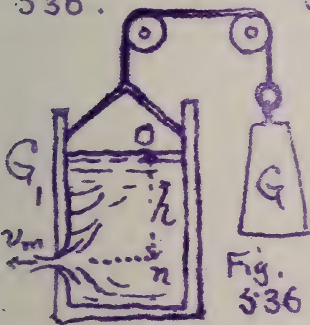


Fig. 536

Suppose the vessel to move upward with a constant acceleration $\bar{p}$. (See § 49a). Taking m and n as in the two preceding §§ we know that $p_m = p'' =$ external pressure $=$ one atmos $= p_a$ [and $\therefore \frac{p_m}{\gamma} = b$. As to the internal pressure at n (same level as m but well back of orifice) p_n, this is not equal to $(b+h)\gamma$, because of the accelerated motion, but we may determine it by considering free the vertical column or prism On of liquid, of cross section $= dF$, the vertical forces acting on which are $p_a\,dF$, downward at O, $p_n\,dF$ upward at n, and its weight, downward, $h\,dF\,\gamma$. All other pressures are horiz. For a vertical upward acceleration $= \bar{p}$, the algebraic sum of the vertical components of all the forces must $=$ mass $\times$ vert. accel, i.e. ...(1.)

$$dF(p_n - p_a - h\gamma) = \frac{h\gamma\,dF}{g}\cdot\bar{p} \;\therefore\; p_n = p_a + h\gamma\left(1+\frac{\bar{p}}{g}\right)$$

which substituted in $v_m = \sqrt{2g\left[\frac{p_n - p_m}{\gamma}\right]}$ (§ 458) gives

$$v_m = \sqrt{2\,(g+\bar{p})\,h} \quad \text{------} \quad (2)$$

It must be remembered that v_m is the velocity of the jet *relatively to the orifice*, which is itself in motion

with a variable velocity. The absolute velocity W_m of the jet is found by the construction in § 83.

If $\bar{p} = g$ = acc. of gravity $v_m = \sqrt{2}\sqrt{2gh}$. If $\bar{p}$ is negative and $= g$, $v_m = 0$ i.e. there is no flow, but both the vessel and its contents fall freely, without mutual action.

Case II. When the liquid and the vessel both have a uniform rotary motion about a vertical axis with angular velocity $= \omega$ (§ 110). Orifice small, so that we may consider the liquid inside (except near the orifice) to be in relative equilibrium. Suppose the jet horizontal at m, Fig. 537, and the radial distance of the orifice from the axis to be $= x$. The external pressure $p' = p_a$, and the internal, see § 410, eqs 3 and 4, is $p_n = p_a + (h_0 + z)\gamma = p_a + h_0\gamma + \frac{\omega^2 x^2}{2g}\gamma$

"BARKER'S MILL"
Fig. 537

∴ veloc. of jet relatively to orifice is (from § 457)

$$v_m = \sqrt{2g\frac{(p_n - p_m)}{\gamma}} = \sqrt{2gh_0 + (\omega x)^2}$$

i.e. $v_m = \sqrt{2gh_0 + \mathrm{W}^2}$ .... (2) in which W, $= \omega x$, = the linear velocity const. of the orifice in its circular path. The absolute velocity W_m *of the particles in the jet* close to the orifice is the diagonal formed on W and v_m (§ 83). Hence by properly placing the orifice in the casing, W_m may be made small or large, and thus the kinetic energy carried away in the effluent water be regulated, within certain limits. Equation (2) will be utilized sub-

sequently in the theory of Barker's Mill.

Example. Let the casing make 100 revol. per min. (whence $\omega = \left[2\pi\ 100 \div 60\right]$ radians per sec.) $h_0 = 12$ feet, and $x = 2$ ft; then (ft. lb. sec.)

$$v_m = \sqrt{2 \times 32.2 \times 12 + \left(\frac{2\pi\ 100 \times 2}{60}\right)^2} = 34.8 \text{ ft. per sec.}$$

(while, if the casing is not revolving, $v_m = \sqrt{2gh_0}$ = only 27.8 " ")

If now the jet is directed horizontally and backward, and also tangentially to the circular path of the centre of the orifice, its absolute veloc. (i.e. relatively to the earth) is $w_m = v_m - \omega x = 34.8 - 20.9 = 13.9$ ft. per sec. and is also horizontal and backwards. The volume of flow being $Q = 0.25$ cub. feet per sec. the kinetic energy carried away with the water per sec. (§ 133) is

$$K.E. = \frac{1}{2} M w_m^2 = \frac{Q\gamma}{g} \cdot \frac{w_m^2}{2} = \frac{\frac{1}{4} \times 62.5}{32.2} \frac{(13.9)^2}{2} = 46.8$$

ft. lbs. per second = 0.085 Horse power.

461. THEORETICAL EFFLUX THROUGH RECTANGULAR ORIFICES OF CONSIDERABLE VERTICAL DEPTH, IN A VERTICAL PLATE. If the orifice is so large vertically that the velocities of the different filaments in a vertical plane of the stream are theoretically different, having different "heads of water", we proceed as follows, taking into account, also, *the velocity of approach*, c, or mean velocity (if any appreciable) of the water in the channel approaching the orifice.

Fig. 538 gives a section of the side of the tank and orifice. Let b = width of the rectangle, the sills of the latter being horizontal, and $a = h_2 - h_1$, its height. Disregarding contraction (for the present), the theoretical volume of discharge per unit of time, =

the sum of the volumes like $v_m\,dF$ ($= v_m\,b\,dx$), in which v_m = the velocity of any filament, as m, in the jet, and $b\,dx$ = cross-section of the small prism which passes through any ~~each~~ horizontal strip of the area of orifice, in a unit of time its altitude being v_m (of prism). For each strip there is a different x or "head of water", and hence a different velocity. Now

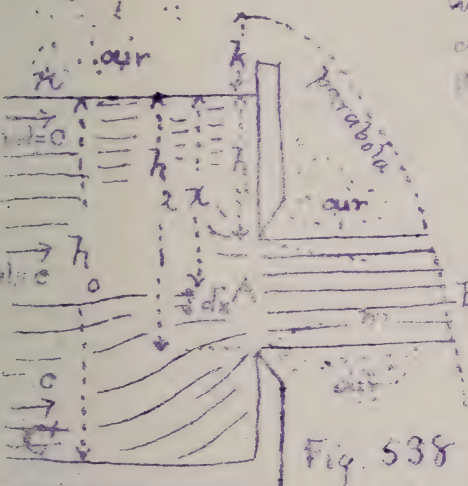


Fig. 538

the theoretical discharge (volume) per unit of time is

$$Q = \left\{ \begin{matrix} \text{sum of the} \\ \text{vols. of all prisms} \end{matrix} \right\} = \int_{x=h_1}^{x=h_2} v_m\,dF = b\int_{h_1}^{h_2} v_m\,dx \quad \ldots (1)$$

But from Bernoulli's theorem, if $k = c^2 \div 2g$ = the velocity-head at n the surface of the channel of approach nC, b being the press. head of n and x its potential-head referred to m as datum, (N.B. This b = 34 ft. and must not be confused with the width b of orifice) we have (see §451 eq. 7)

$$\frac{v_m^2}{2g} + b + 0 = \frac{c^2}{2g}(\text{or } k) + b + x, \quad \therefore \quad v_m = \sqrt{2g}\,\sqrt{x+k} \quad \ldots (2)$$

and since dx also $= d(x+k)$, taking $(x+k)$ as the variable we have from (1)

$$\text{Theoret. } Q = \underset{\text{width}}{b}\sqrt{2g}\int_{h_1+k}^{h_2+k} (x+k)^{1/2}\,d(x+k) \quad \ldots \text{which gives}$$

RECT. ORIF. in THIN PLATE

$$\text{Theoret. } Q = \frac{2}{3}\,b\sqrt{2g}\left[(h_2+k)^{\frac{3}{2}} - (h_1+k)^{\frac{3}{2}}\right] \quad \ldots (3)$$

If c is small, the channel of approach being large we have Th^{al.} $Q = \frac{2}{3} b \sqrt{2g} \left(h_2^{3/2} - h_1^{3/2} \right)$ ..... (2)

c being = $Q \div$ area of section of nC

If $h_1 = 0$ i.e. if the orifice becomes a **NOTCH IN THE SIDE**, or an **OVERFALL** [see Fig. 539 which shows the contraction, which actually occurs in all these cases] we have for an

width

OVERFALL .... Theord. $Q = \frac{2}{3} b \sqrt{2g} \left[(h_2 + k)^{3/2} - k^{3/2} \right]$.(3)

Fig. 539.

NOTE. Both in (1) and (2) h_1 and h_2 are the vertical depths of the respective sills of the orifice from the surface of the water three or four feet back of the plane of the orifice where the surface is comparatively level. This must be specially attended to in deriving the actual discharge from the theoretical (see § 463). Since the velocity of approach, involved in $k = c^2 \div 2g$, depends on Q, (thus: $F_0 c = Q$ or $c = Q \div F_0$ ..... (3) where F_0 is the area of cross-section of the channel of approach nC

If Q is the unknown quantity, it is necessary to proceed by successive assumptions and approximations in using eq.s (1)(2) and (3).

With $k = 0$, (or c very small i.e. F_0 very large) eq. (3) reduces to

width

OVERFALL ......Q (theoret.) $= \frac{2}{3} b h_2 \sqrt{2g h_2}$ .... ($3\frac{1}{2}$)

or $\frac{2}{3}$ as much as if all parts of the orifice had the same head of water $= h_2$ (as for instance if the orifice were in the horizontal bottom of a tank in which the water was h_2 deep the orifice being b by h_2)

THEORET. EFFLUX THRO' A TRIANGULAR ORIFICE IN A THIN VERTICA PLATE OR WALL. BASE horizontal. Fig. 540. Let the channel of approach be so large that the velocity of approach may be neglected. h_1 and h_2 = depths of sill and vertex, which is downward. The analysis differs from that of the preceding § only in having $k = 0$ and the length u, of a horiz. strip of the orifice, variable b being the length of the base of the triangle, we have from similar triangles

$$\frac{u}{b} = \frac{h_2 - x}{h_2 - h_1} \quad \text{i.e. } u = \frac{b}{h_2 - h_1}(h_2 - x)$$

Fig. 540

Theoret. $Q = \int v_m \, dF = \int v_m u \, dx$

$$= \frac{b}{h_2 - h_1} \int_{h_1}^{h_2} v_m (h_2 - x)\, dx$$

and finally, see eq. (2') of § 461 we have

$$\left.\begin{matrix}\text{Theoret}\\ Q\end{matrix}\right\} = \frac{b\sqrt{2g}}{h_2 - h_1}\int_{h_1}^{h_2}(h_2 - x)\,x^{1/2}\,dx = \frac{2}{15}\,\frac{b\sqrt{2g}}{h_2 - h_1}\left[2h_2^{5/2} - 5h_2 h_1^{3/2} + 3h_1^{5/2}\right]$$

For a triangular notch or overfall as in Fig. 541, this reduces to

$$\left.\begin{matrix}\text{Theoret}\\ Q\end{matrix}\right\} = \frac{4}{15}\, b h_2 \sqrt{2gh_2} = \frac{8}{15}\,\frac{b h_2}{2}\sqrt{2gh_2} \quad \ldots\ldots (5')$$

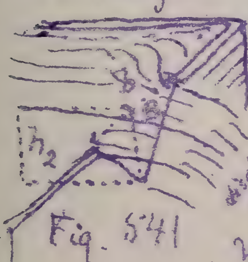


Fig. 541

i.e. $\frac{8}{15}$ the volume discharged per unit of time if the orifice with base b and altitude h_2 had been in the horizontal bottom of a tank under a head of h_2. The measurement of h_2 and b are made with reference to the level surface back of the orifice see figure.

for the water surface in the plane of the orifice is curved below the level surface of the tank.

Prof. Thomson has found by experiment that with $b = 2h_2$, the actual discharge = theoret. disch. $\times$ 0.595; and with $b = 4h_2$ actual = " " $\times$ 0.620.

463. ACTUAL DISCHARGE THRO' SHARP-EDGED RECTANGULAR ORIFICES (sills horizontal) in the vertical side of a tank or reservoir. h_1 not zero; i.e. no overfall.

Case I. Complete ~~and~~ Perfect Contraction. The actual volume of water discharged per unit of time is much

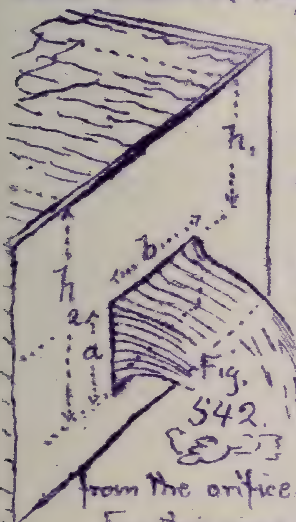


Fig. 542.

less than the theoretical values ~~derived~~ in § 461, chiefly on account of contraction. By complete contraction we mean that no edge of the orifice is flush with the side or bottom of the reservoir; ~~and by perfect contrac-~~ tion that the channel of approach to whose surface the heads h_1 and h_2 are measured is so large that the contraction is practically the same in effect as if the channel were of infinite extent sideways and downward from the orifice.

For this case (h_1 not zero) it is found most convenient to use the following practical formula (b = width)

$$\left.\begin{array}{l}\text{Vol. of actual}\\ \text{disch. per time unit}\end{array}\right\} = Q = \mu_0 \, ab\sqrt{2g\left[h_1 + \frac{a}{2}\right]} \quad \ldots (6.)$$

in which (see Fig. 542) a = the height of orifice, h_1 the vertical depth of the upper edge of the orifice below the level of the reservoir surface measured a few feet back of the plane of the orifice, and μ_0 a coefficient of efflux, dependent on experiment, and an abstract number.

With $\mu_0 = 0.62$ approximate results (within 3

or 4 %) may be obtained from eq. 6 with openings not more than 18 inches or less than 1 inch high, and not less than 1 inch wide, with heads $(h_1 + \frac{a}{2})$ from 1 ft. to 20 or 30 feet.

Example. What is the actual discharge (volume) per ~~minute through~~ the orifice in Fig. 542, 14 in. wide and one foot high, the upper sill being 8 ft 6 in. below the surface of still water. Use eq. (6) with the ft. lb. ~~and~~ sec. as units and $\mu_0 = 0.62$.

Solution:

$$Q = 0.62 \times 1 \times 1\tfrac{1}{6} \times \sqrt{2 \times 32.2\left[8\tfrac{1}{2} + \tfrac{1}{2}\right]} = 17.46 \text{ cub. ft. per sec.}$$

while the flow of weight is

$$G = Q\gamma = 17.46 \times 62.5 = 1091. \text{ lbs per second}$$

For comparatively accurate results, values of μ_0, taken from the following table (computed from the careful experiments of Poncelet and Lesbros) may be used for the sizes there given, and where practicable, for other sizes by interpolation. To use the table, the values of h_1, a, and b must be reduced to metres which can be done by the reduction table below; but in substituting in eq. 6, if the metre-kilogram-second system of units be used g must be put $= 9.81$ metres per square second (see § 51) and Q will be obtained in cub. metres per sec:

TABLE FOR REDUCING FEET AND IN.S TO METRES

1 foot =	0.30479 met.s	1 inch =	0.0253 metres
2 feet =	0.60959 " "	2 inches =	0.0507 "
3 " =	0.91438 " "	3 " =	0.0761 "
4 "	1.21918 "	4 "	0.1015 "
5 "	1.52397 "	5 " =	0.1268 "
6 "	1.82877 "	6 "	0.1522 "
7 "	2.13356 "	7 " =	0.1776 "
8 "	2.43836 "	8 "	0.2030 "
9 " =	2.74315 "	9 " =	0.2283 "
10 " =	3.04794 "	10 "	0.2536 "
		11 " =	0.2790 "

VALUES OF μ_0 (for eq. 6) for RECT. ORIF. THIN PL.

Value of h_1, Fig. 542, in metres	$b = .20^m$ $a = .20^m$ (relative size)	$b = .20^m$ $a = .10$	$b = .20^m$ $a = .05$	$b = .20^m$ $a = .03$	$b = .20^m$ $a = .02$	$b = .20^m$ $a = .01$
	μ_0	μ_0	μ_0	μ_0	μ_0	
0.005^m						0.705
0.010			0.607	0.630	.660	.701
0.015		0.593	.612	.632	.660	.697
0.020	0.572	.596	.615	.634	.659	.694
0.030	0.578	.600	.620	.638	.659	.688
0.040	.582	.603	.623	.640	.658	.683
0.050	.585	.605	.625	.640	.658	.679
0.060	.587	.607	.627	.640	.657	.676
0.070	.588	.609	.628	.639	.656	.673
0.080	.589	.610	.629	.638	.656	.670
0.090	.591	.610	.629	.637	.655	.668
0.100	.592	.611	.630	.637	.654	.666
0.120	.593	.612	.630	.636	.653	.663
0.140	.595	.613	.630	.635	.651	.660
0.160	.596	.614	.631	.634	.650	.658
0.180	.597	.615	.630	.634	.649	.657
0.200	.598	.615	.630	.633	.648	.655
0.250	.599	.616	.630	.632	.646	.653
0.300	.600	.616	.629	.632	.644	.650
0.400	.602	.617	.628	.631	.642	.647
0.500	.603	.617	.628	.630	.640	.644
0.600	.604	.617	.627	.630	.638	.642
0.700	.604	.616	.627	.629	.637	.640
0.800	.605	.616	.627	.629	.636	.637
0.900	.605	.615	[illegible]	.628	.634	.635
1.000	.605	.615	.626	.628	.633	.632

continued on next page

VALUES OF μ_0 (for eq. 6) for RECT. ORIF. in thin plate (continued)

Value of h_1, Fig. 542 in metres	$b = .20^{met.}$ $a = .20^{m}$	$b = .20^{m}$ $a = .10^{m}$	$b = .20^{m}$ $a = .05^{m}$	$b = .20^{m}$ $a = .03$	$b = .20^{m}$ $a = .02$	$b = .20^{m}$ $a = .01$
1.100	.604	.614	0.625	.627	.631	.629
1.200	.604	.614	.624	.626	.628	.626
1.300	.603	.613	.622	.624	.625	.622
1.400	.603	.612	.621	.622	.622	.618
1.500	.602	.611	.620	.620	.619	.615
1.600	.602	.611	.618	.618	.617	.613
1.700	.602	.610	.617	.616	.615	.612
1.800	.601	.609	.615	.615	.614	.612
1.900	.601	.608	.614	.613	.612	.611
2.000	.601	.607	.613	.612	.612	.611
3.000	.601	.603	.606	.608	.610	.609

Continued on p. 119 for two wider orifices.

Example. With $h_1 = 4$ in. ($= 0.10^{met}$); $a = 8^{in.}$ ($= 0.20$ met), $b = 1$ ft. 8 in. ($= 0.51^{met.}$), required the volume (actual) discharged per sec. Com. & perf. contr.

From the table (see also p. 119) we have

for $h_1 = 0.10$ m., $b = 0.60$ m. and $a = 0.20$ m.; $\mu_0 = .602$
for $h_1 = 0.10$ m, $b = 0.20$ m, " $a = 0.20$ m; $\mu_0 = .592$
$\therefore$ for $h_1 = 0.10$, $b = 0.51$ m, " $a = 0.20$ we $\overline{.010}$
shall have by interpolation

$$\mu_0 = 0.602 - \frac{9}{40}\left[.602 - 0.592\right] = 0.600$$

Hence (ft. lb. sec., remembering μ_0 is an abst. numb.) from eq.(6)

$$Q = 0.600 \times \frac{8}{12} \times \frac{20}{12} \sqrt{2 \times 32.2\left(\frac{4}{12} + \frac{1}{2}\cdot\frac{8}{12}\right)} = 0.544 \left\{\begin{matrix}\text{cub.}\\ \text{ft.}\\ \text{per}\\ \text{sec.}\end{matrix}\right.$$

VALUES OF μ_0 for rect. orif. in thin plate (continued)

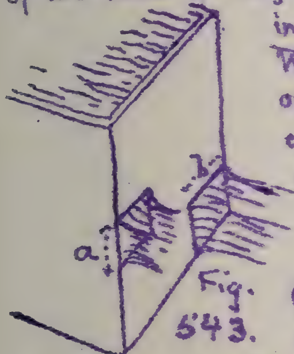
Value of h_1, Fig 542	$b = .60^m$ $a = .20^m$	$b = 0.60^m$ $a = .02$
	$\mu_0 =$	
0.010^{met}		0.644
0.015		.644
0.020		.643
0.030	0.593	.642
0.040	0.595	.642
0.050	0.597	.641
0.060	0.599	.641
0.070	0.600	.640
0.080	.601	.640
0.090	.601	.639
.100	.602	.639
.120	.603	.638
.140	.603	.637
.160	.604	.637
.180	.605	.636
.200	.605	.635
.250	.606	.634
.300	.607	.633
.400	.607	.631
.500	.607	.630
.600	.607	.629
.700	.607	.628
.800	.606	.628
.900	.606	.627
1.000	.605	.626

Value of h_1 Fig 542	$b = .60$ $a = .20^m$	$b = .60$ $a = .02^m$
1.100	.604	.626
1.200	.604	.625
1.300	.603	.624
1.400	.603	.624
1.500	.602	.623
1.600	.602	.623
1.700	.602	.622
1.800	.602	.621
1.900	.602	.621
2.000	.602	.620
3.000	.601	.615

Case II. Incomplete Contraction. This name is given to the cases, like the one shown in Fig. 543, where one or more sides of the orifice have an interior border, i.e. are flush with the sides or bottom of the tank (square-cornered).

Not only is the general direction of the stream altered but the discharge is greater on account of the larger size of the contracted section, since

contraction is prevented on those edges which have a border. It is assumed that the ~~partial~~ contraction which does occur (on the other edges) is *perfect*, i.e. the cross-section of the tank is large compared with the orifice. According to the experiments of Bidone and Weisbach, with Poncelet's orifices (i.e. orifices in thin plate mentioned in the preceding Case I.) the actual volume discharged per unit of time is



$$Q = \mu\, ab\sqrt{2g\left(h_1 + \frac{a}{2}\right)} \quad \cdots\cdots (7)$$

Fig. 543.

(differing from eq. (6) only in the co-efficient of efflux μ) ~~in which the abstract number~~ μ is thus found: ~~Determine a co-ef~~ficient μ_0 as if ~~eq. (6) were~~ to be used, i.e., as if the contraction were complete ~~and perfect~~ (Case I.), then

$$\mu = \mu_0\left[1 + 0.155\, n\right] \quad \cdots\cdots (7)'$$

where n = the ratio of the length of of ~~periphery of the~~ orifice with a border to the whole periphery.

E.g. if the lower sill, only, has a border, $n = b \div 2(a+b)$

" " " " " " and both sides " " " $n = (2a+b) \div 2(a+b)$

Example. If $h_1 = 8^{\text{ft}}$ $(= 2.43^{\text{met.}})$, $b = 2^{\text{ft.}}$ $(= 0.60^{\text{m.}})$, $a = 4$ in. $(= 0.10$ met.$)$, and one side is even with the side of the tank, and the lower sill even with the bottom, required the volume discharged per second. (Sharp-edged orifice, in vertical plane, etc.)

Here for comp. and perf. contraction we have from Poncelet's tables, (Case I) $\mu_0 = 0.608$. Now $n = \frac{1}{2}$; $\therefore$ from eq. (7)', $\mu = 0.608\left[1 + 0.155 \times \frac{1}{2}\right] = 0.6551$

$\therefore$ eq. 7, $Q = .655 \times 2 \times \frac{4}{12}\sqrt{2 \times 32.2\left(8 + \frac{1}{2}\cdot\frac{4}{12}\right)} =$ 12.23 cub. ft per sec.

Case III. Imperfect Contraction. If there is a submerged channel of approach symmetrically placed as regards the orifice, and of an area (cr. section) $= G$, not much larger than that $= F$, of the orifice (see Fig. 544) the contraction is less than in Case I, and is called imperfect contraction. Upon his experiments with Poncelet's orifices, with imperfect contraction Weisbach base the following formula for the discharge (vol.) per unit of time, viz.:

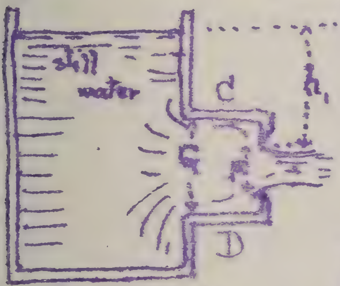


Fig. 544.

$$Q = \mu a b \sqrt{2g\left(h_1 + \frac{a}{2}\right)} \quad \ldots\ldots (8)$$

(see Fig. 542 for notation) with the understanding that the co-efficient $\mu = \mu_0 (1 + \beta) \quad \ldots\ldots (8)'$

where μ_0 is the co-efficient obtained from the tables of Case I (as if the contraction were perfect and complete) and β an abstract number depending on the ratio $F : G = m$, as follows

$$\beta = 0.0760\left[9^m - 1.00\right] \quad \ldots (8)''$$

TABLE A.

m	β	m	β
.05	.009	.55	.178
.10	.019	.60	.208
.15	.030	.65	.241
.20	.042	.70	.278
.25	.056	.75	.319
.30	.071	.80	.365
.35	.088	.85	.416
.40	.107	.90	.473
.45	.128	.95	.537
.50	.152	1.00	.608

Example. Let $h_1 = 4' 9\frac{1}{2}''$ ($= 1.46$ met.), the dimensions of the orifice being width $= b = 8$ in. ($= 0.20^{m.}$) height $= a = 5$ in ($= 0.126^{m.}$) while the channel of approach (CD, Fig. 544) is one foot square. From Case I, we have, for the given dimensions and head,

$\mu_0 = 0.610$ Since

$\frac{F}{G} = \frac{40 \text{ sq. in}}{144 \text{ sq. in}} = 0.27$ we find [Table A] $\beta_1 = 0.062$
and hence $\mu = \mu_0 (1.062)$

$\left.\begin{matrix}\therefore \text{eq}(8) \\ \text{ft. lb. sec.}\end{matrix}\right\} Q = 0.610 \times 1.062 \times \frac{5}{12} \cdot \frac{8}{12} \sqrt{2 \times 32.2 \times 5} = \left\{\begin{matrix}3.22 \\ \text{cub. ft} \\ \text{per sec.}\end{matrix}\right.$

Case IV. *Head measured in moving water.*

Fig. 545. If the head h_1, of the upper sill, can not be measured to the level of still water, but must be taken to the surface of a channel of approach, where the velocity of approach is quite appreciable, not only [illegible] contraction imperfect, but strictly we should use eq. (1) of § 461, in which the velocity of approach is considered.

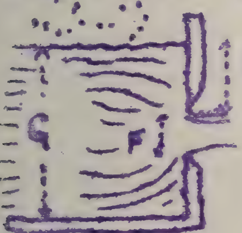


Fig. 545.

Let F = area of orifice, and G that of the cross-section of the channel of approach; then the vel. of approach is $c = Q \div G$, and k (of above eq.) $= c^2 \div 2g$ $= Q^2 \div 2g\,G^2$; but Q itself being unknown, a substitution of k in terms of Q in eq. (1), § 461, leads to an equation of high degree with respect to Q. Practically therefore it is better to write

$$Q = \mu a b \sqrt{2g\left(h_1 + \frac{a}{2}\right)} \text{ - - - - - } (9.)$$

and determine μ by experiment for different values of the ratio $F \div G$. Accordingly, Weisbach found, for Poncelet's orifices, that if μ_0 is the co-efficient for complete and perf. contraction from Case I, we have

$$\mu = \mu_0 (1 + \beta') \text{ - - - - - } (9)'$$

β' being an abstract number, and being thus related to $F \div G$: $\beta' = 0.641 \left(\frac{F}{G}\right)^2$ .....(9)"

h_1 was measured to the surface one meter back of the plane of the orifice, and $F : G$ did not exceed 0.50. He gives the following table computed from eq. (9):

TABLE B

$F \div G$	β'
0.05	.002
.10	.006
.15	.014
.20	.026
.25	.040
.30	.058
.35	.079
.40	.103
.45	.130
.50	.160

Example. A rectangular water-trough 3 ft. wide is dammed up with a vertical board in which is a rectangular orifice, as in Fig. 545, of width $b = 2$ ft ($= 0.60^{met.}$) and height $a = 6^{in}$ ($= 0.15^{m.}$); and when the water-level behind the board has ceased rising (i.e. when the flow has become steady) we find that $h_1 = 2$ ft. and the depth behind in the trough to be 3 ft. Required Q.

Since $F : G = 1^{sq.ft} \div 12^{sq.ft} =$ $= .0883$ we find (Table B) $\beta' = 0.005$ and μ_0 being $= 0.612$ from Poncelet's Tables Case I we have finally from eq. (9)

$$Q = 0.612(1.005)2 \times \tfrac{1}{2}\sqrt{2 \times 32.2 \times 2.25}$$

$= 7.41$ cub. ft. per second.

464. ACTUAL DISCHARGE OF SHARP-EDGED OVERFALLS (OVERFALL-WEIRS; OR rectang. NOTCHES IN A THIN VERTICAL PLATE.

Case I. *Complete and Perfect Contraction* (the normal case), Fig. 546; i.e. no edge is flush with the side or bottom of the reservoir, whose cross section is very large compared with $b\,h$ of the notch.

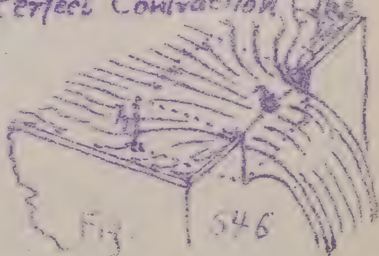


Fig. 546

By depth, h_2, of the notch we are to understand the depth of the sill below the surface a few feet back of the notch where it is level. In the plane of the notch the vertical thickness of the stream is only from 3/4 to 9/10 of h_2. Putting ∴ the velocity of approach = zero, and ∴ $k = 0$, in eq. (3) of § 461, we have for the

ACTUAL DISCH., $$Q = \mu_0 \frac{2}{3} b h_2 \sqrt{2 g h_2} \quad \text{---(10)}$$

(b = width of notch), where μ_0 is a co-efficient of efflux to be determined by experiment.

Experiments with overfalls do not agree as well as might be desired. Those of Poncelet and Lesbros gave the results in Table C.

TABLE C

For $b = 0.20^m$		For $b = .60^m$	
h_2 (metres)	μ_0	h_2 (met.)	μ_0
.01 m.	.686	.06	.618
.02	.620	.08	.613
.03	.618	.10	.609
.04	.610	.12	.605
.06	.601	.15	.600
.08	.595	.20	.592
.10	.592	.30	.586
.15	.589	.40	.586
.20	.585	.50	.596
.22	.577	.60	.585
metres		metres	

For approx. results $\mu_0 = .60$

Example 1. With $h_2 = 1$ ft. 4 in. ($= .405^m$) $b = 2^{ft}$ ($= 0.60^m$) we have from Table C $\mu_0 = .586$

∴ Q ..... (ft. lb. sec.)

$= .586 \times \frac{2}{3} \times 2 \times \frac{4}{3} \sqrt{2 \times 32.2 \times \frac{4}{3}}$

$= 9.64$ cub. ft. per sec.

Example 2. What width b, must be given to a rectangular notch, for which $h_2 = 10$ in. ($= 0.25$ m.), that the discharge may be $Q = 6$ cub. feet per sec.?

Since b is unknown we can not use the table immediately but take $\mu_0 = .600$ for a first approximation, whence, eq. 10, ft. lb. sec.,

$$b = 6 \div \left[0.6 \times \frac{2}{3} \times \frac{10}{12} \sqrt{2 \times 32.2 \times \frac{10}{12}}\right] = 2.46 \text{ ft.} = .75 \text{ met.}$$

Then since this width does not much exceed 0.60 met. we may take, in Table C, for $h_2 = .25$ m., $\mu_0 = .589$

$$\therefore b = \frac{6}{.589 \times \frac{2}{3} \times \frac{10}{12}\sqrt{2 \times 32.2 \times \frac{10}{12}}} = 2.50 \text{ feet}$$

Ans.

Case II. Incomplete Contraction, i.e. both ends are flush with the sides of the tank, these being ⟂ to the plane of the notch. According to Weisbach we may write

$$Q = \frac{2}{3}\mu b h_2 \sqrt{2gh_2} \quad \ldots\ldots (11)$$

in which $\mu = 1.0615\,\mu_0$, μ_0 being obtained from Table C for the normal case, Case I. The section of channel of approach is large compared with that of notch; if not, see Case IV.

Case III. Imperfect Contraction, i.e., the velocity of approach is appreciable; the sectional area G of the channel of approach not being much larger than $F = bh_2$ = area of notch. Fig. 547. b = width, h_2 = height (see Fig. 546) of notch.

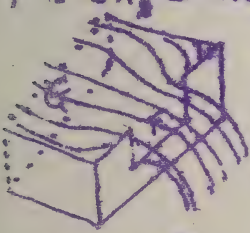


Fig. 547

Here instead of using a formula involving $k = c^2 \div 2g = [Q \div G]^2 \div 2g$ (see eq. 3 § 461), it is more to convenient to put

$$Q = \frac{2}{3}\mu b h_2 \sqrt{2gh_2} \quad \ldots\ldots (12)$$

as before; with $\mu = \mu_0(1+\beta) \ldots (12)'$

in which μ_0 is for the normal case, Case I, and β, according to Weisbach's experiments, may be obtained from the empirical formula

{Table D is computed from this} $\beta = 1.718\left(\frac{F}{G}\right)^4 \quad \ldots\ldots (12)''$

[The end-contraction is complete in this case, i.e. the ends are not flush with the sides of tank]

TABLE D

$F:G$	β
0.05	.000
.10	.000
.15	.001
.20	.003
.25	.007
.30	.014
.35	.026
.40	.044
.45	.070
.50	.107

Example. If the water in the channel of approach has a vertical section (transverse) of $G = 7$ sq. ft. while the notch is 2 ft wide $= b$ and one foot deep (i.e., $h_2 = 1^{ft}$ to surface of the level water behind) we have from table C, with $b = .60$ metres and $h_2 = 0.30$ met.,

$\mu_0 = 0.586$, while from table D, with $F \div G = 0.222$ (or $\frac{2}{7}$) $\beta = .005$ ∴ (ft. lb. sec.)

$$Q = \frac{2}{3} \times .586 \times 1.005 \times 2 \times 1 \sqrt{64.4 \times 1.0}$$

$= 6.30$ cub ft per second

Case IV. Imperfect and incomplete contraction together; both end-contractions being suppressed (by making the ends flush with the sides of the reservoir, these sides being vertical and ⊥ to the plane of the notch) and the channel of approach not being very deep, i.e., having a sectional area G not much larger than that, F, of notch. $F = bh_2$ as before. Again we write

$$Q = \frac{2}{3}\mu b h_2 \sqrt{2g h_2} \quad \text{-------} \; 13$$

with μ computed from $\mu = \mu_0 (1 + \beta) \ldots\ldots (13)'$

μ_0 being obtained from table C, while

$$\beta = 0.041 + 0.3693 \left(\frac{F}{G}\right)^2 \ldots (13)''$$

an empirical formula based by Weisbach on his own experiments.

Table E is computed from $(13)''$ (see next page).

Example. Fig. 547a. With $b = 2' = (0.60^m)$ and $h_2 = 1'$ ($= 0.30^{met.}$), we have from table C $\mu_0 = 0.586$. But, the ends being flush with the

TABLE E

$F \div G$	β
0.00	0,041
.05	,042
.10	,045
.15	.049
.20	.056
.25	.064
.30	,074
.35	,086
.40	.100
.45	.116
.50	0.133

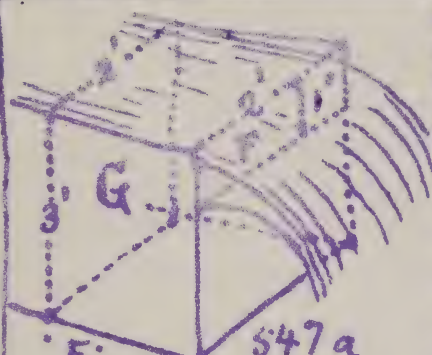


Fig. 547 a.

with the sides of the tank, and $G = 6$ sq. feet, not being excessively larger than $F = 2$ sq. ft., we have from table E, with $F \div G = .333$, $\beta = .081$, and ∴, eq. (13) and (13)', (μ_0 being as in last example)

$$Q = \frac{2}{3} \times 0.586 \times (1 + .081)\, 2 \times 1 \sqrt{64.4 \times 1} = 6.98 \left\{ \begin{array}{l} \text{cub.} \\ \text{ft.} \\ \text{per} \\ \text{sec.} \end{array} \right.$$

465. FRANCIS FORMULA FOR OVERFALLS; (RECTANGULAR).

From the experiments at Lowell, Mass., in 1851, with overfall weirs Mr. J. B. Francis deduced the following formula for the volume, Q, of flow per second over rectang. weirs 10 feet in width, and with h_2 = from 0.6 to 1.6 feet (from sill to level surface of water a few feet back) (b = width)

$$Q = \frac{2}{3} \times 0.622 \left(b - \frac{1}{10} n h_2 \right) \sqrt{2 g h_2} \ \ldots \left\{ \begin{array}{l} \text{NOT HOMOG.;} \\ \text{ft. lb. sec.} \\ \ldots \ (14) \end{array} \right.$$

which provides for incomplete contraction as well as for complete and perfect contraction by making

$n = 2$ for perfect and complete contraction.

$n = 1$ when one end only is flush with side of channel

$n = 0$ " both ends are " " sides " ".

466. FTELEY AND STEARNS'S EXPERIMENTS

at Boston, Mass., in '77 & '80 are given in the Transactions of the Am. Soc. Civ. Engineers, Vol. XII, and produced formulae slightly different from those of Mr. Francis in some particulars. In the case of imperfect and incomplete contraction, like that in Fig. 547 a, for example, they propose formulae as follows:

(See interleaved Hektograph sheet)

NOTE. The works of Trautwine, Jackson, Weisbach and Rankine give numerous formulae for use in Hydraulics.

466. SHORT CYLINDRICAL TUBES. When efflux takes place through a short cylindrical tube, at least 2½ times as long as wide, inserted at right angles in the plane side of a large reservoir, Fig. 548, the jet issues from the end of the tube in parallel filaments, and a sectional area F_m equal to F that of the tube. [To attain this result, however, the tube must be full of water before the outer end is unstopped], and must not be oily; also the head h must not be > 40 ft for efflux into the air] Since at m

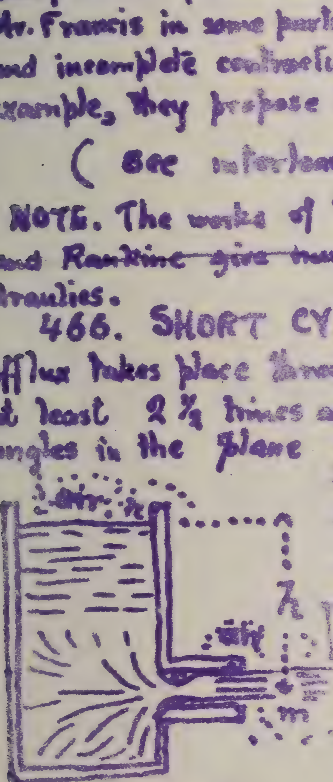


Fig. 548.

the filaments are parallel and the pressure-head ∴ equal to b (= 34 ft. for water) = that of surrounding medium = one atmosphere in this instance, an application of Bernoulli's Theorem (eq. 7 § 451) to positions m and n would give us (precisely as in §§ 454 and 455) v_m = veloc. at m = $\sqrt{2gh}$, theoretically; but experiment shows that the actual value of v_m in this case is $v_m = \phi_0\sqrt{2gh} = .815\sqrt{2gh}$ ..(1)

.815 being a close average for ϕ_0, the co-efficient of velocity, for ordinary purposes. It increases slightly as the head decreases, and is evidently much less than the value 0.97 for an orifice in a thin plate, § 454, or for a rounded mouthpiece as in § 455.

But as the sectional area of the stream where the filaments are parallel, at m, where $v_m = .815\sqrt{2gh}$ is also equal to that, F, of the tube, the co-efficient of efflux, μ_0, in the formula $Q = \mu_0 F \sqrt{2gh}$ is $= \phi_0$ i.e. there is no contraction and the C of § 454 = 1.00.

Hence for the volume of discharge per unit of time, we have practically,

$$Q = \phi_0 F\sqrt{2gh} = 0.815\, F\sqrt{2gh} \quad \ldots .(2)$$

The discharge is ∴ about $\frac{1}{3}$ greater than through an orifice of the same diameter in a thin plate under the same head (compare eq. (3) § 454), for although the velocity is less at m in the present case the section of the stream at m is greater, there being no contraction (at m).

This difference in velocity is due principally to the fact that the entrance of the tube has square edges so that the stream contracts (at m' Fig. 549) to a section smaller than that of the tube, and then re-expands to the full section F, of tube.

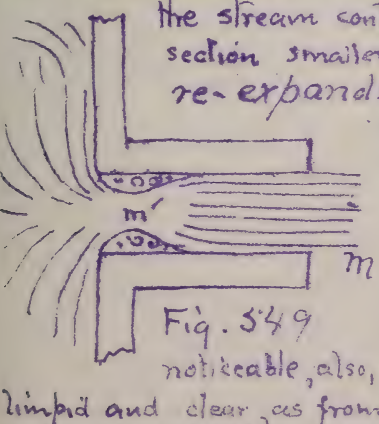


Fig. 549

The eddying and accompanying internal friction caused by this re-expansion, (or "sudden enlargement" of the stream) is the principal resistance which diminishes the velocity. It is noticeable, also, in this case that the jet is not limpid and clear, as from thin plate, but troubled and only translucent (like ground-glass). The internal press-

ure in the stream at m' is found to be less than one atmosphere, i.e. < that at m, as shown experimentally by the sucking in of air when a small aperture is made in the tube opposite m'. If the tube itself were formed internally to fit this contracted vein, as in Fig. 550, the eddying would be prevented and the slight diminution of velocity, as compared with $\sqrt{2gh}$, then produced would not differ greatly from that occurring with an orifice in thin plate or the orifice of § 455, being due chiefly to surface friction.

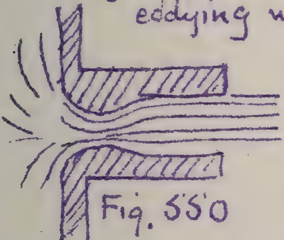


Fig. 550

If the tube is less than $2\frac{1}{2}$ times as long as wide, or if the interior is not wet by the water (as when oily) or if the head is > about 40 ft. the efflux takes place as if the tube were not there, Fig. 551, and we have

$$v_m = 0.97\sqrt{2gh}, \text{ as in § 45?}$$

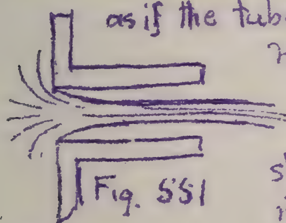


Fig. 551

Example. The discharge thro' a short pipe, 3 in. in diameter, like that in Fig. 548, is 30 cub. feet per minute, under a head of 2 ft 6 in. Reservoir large. Required the co-efficient of efflux $\mu_o = \phi_o$ in this case. For variety use the *inch-lb.-minute* system of units in which $g = 32.2 \times 12 \times 3600$, (see NOTE, § 51) μ_o being an abstract number will be the same, numerically, whatever the system of units. From eq.(2)

$$\phi_o = \mu_o = \frac{Q}{F\sqrt{2gh}} = \frac{30 \times 1728}{\frac{\pi}{4} \times 3^2 \sqrt{2 \times 32.2 \times 12 \times 60^2 \times 30}}$$

$= .803$

467. INCLINED SHORT TUBES (CYLINDRICAL Fig. 552. If the short tube is inclined at some angle

$\alpha < 90°$ to the interior plane of the reservoir wall,

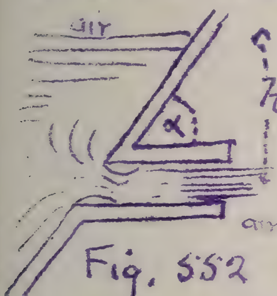


Fig. 552

the efflux is smaller than when the angle is 90°, as in § 466. We still use the form of equation

$$Q = \mu F \sqrt{2gh} = \phi F \sqrt{2gh} \quad \ldots (3)$$

but from Weisbach's experiments should take μ from the following

TABLE

For $\alpha =$	90°	80°	70°	60°	50°	40°	30°
$\mu = \phi =$	.815	.799	.782	.764	.747	.731	.719

Example. With $h = 12$ ft. d = diam. = 4 in. and $\alpha = 46°$ we have the discharge (ft. lb. sec.)

$$Q = \left[0.731 + \frac{6}{10}(.016)\right]\frac{\pi}{4}\left(\frac{1}{3}\right)^2\sqrt{64.4 \times 12} = 1.79 \left\{\begin{matrix}\text{cub.}\\ \text{ft}\\ \text{per}\\ \text{sec.}\end{matrix}\right.$$

468. DIVERGING AND CONVERGING SHORT TUBES. With conically convergent tubes, as at A,

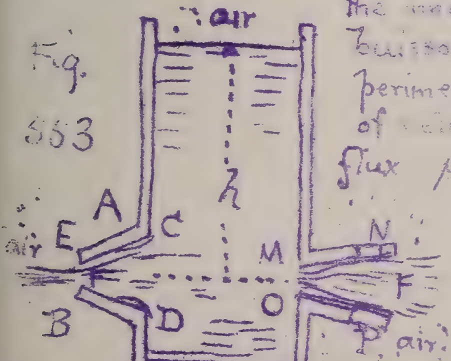


Fig. 553

the inner edge not rounded, D'Aubuisson and Castel found, by experiment, values of the co-efficient of velocity ϕ, and that of efflux μ (from which the co-efficient $C = \mu \div \phi$, of contraction, may be computed) for tubes 1.55 centims. wide at the narrow end, and 4 cent's long, under a head of 3 meters, and different angles of convergence. By angle of convergence is meant the angle between the sides CE and DB Fig. 553, and the value of μ taken from the following table

F, is to be substituted in the formula $Q = F\mu\sqrt{2gh}$, where F denotes the area of the outlet orifice EB. Evidently (see table) μ is a max. for $13\frac{1}{2}°$

TABLE F

Ang. of conv. =	3° 10',	8°,	10° 20',	13° 30',	19° 30',	30°,	49°
μ =	.895	.930	.938	.946	.924	.895	.847
ϕ =	.894	.932	.951	.963	.970	.975	.98[illegible]

With a conically divergent tube as at MN, with the internal diam. MO = .026 met., the internal diam. NP = .032 met., and the angle between MN and PO = 4° 50', [illegible] found that in the equation $Q = \mu F\sqrt{2gh}$, where F = area of outlet section NP, μ to be = [illegible], the great loss of velocity, as compared with the [illegible] $\sqrt{2gh}$, being due to the eddying in the re expansion from the contracted section at M (corners not rounded), as occurs also in Fig. 549. The jet was much [illegible] and pulsated violently. When the angle of divergence is too great, or the head h too large, or if the tube is not [illegible] by the water, efflux with the tube filled cannot be secured, the flow taking place as in Fig. [illegible], just as for a straight tube.

Venturi and Eytelwein experimented (called "Venturi's tube") with a conically divergent tube, with rounded entrance to conform to the shape of the contracted vein as in Fig. 55[illegible] having a diameter of [illegible] at m' (narrowest place) where

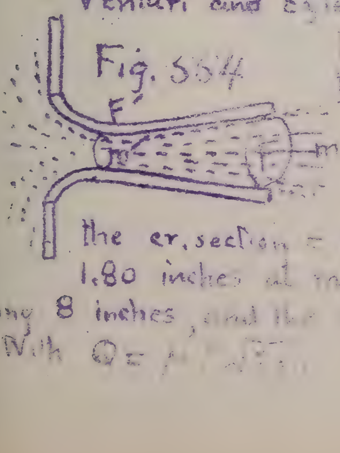


Fig. 554

the cr. section = F' = 0.7854 sq. in., and of 1.80 inches at m (outlet area = F), the length being 8 inches, and the angle of divergence [illegible]. With $Q = \mu F\sqrt{2gh}$ they found μ = [illegible]

Hence it discharged 2.4 times as much water as would have flowed under the same head thro' an orifice in thin plate with area = F' = the smallest section of the diverging tube, and 1.9 times as much as through a short pipe of section = F'. A similar calculation shows that the velocity at m' must have been = $1.55\sqrt{2gh}$ and the pressure was much less than atmospheric.

J. B. Francis also experimented with Venturi's tube, (see "Lowell Hydraulic Experiments".)

469. FRICTION BETWEEN LIQUIDS AND THE SURFACES OF SOLIDS. In long pipes the friction (or adhesion) of the liquid against the inner surface of the pipe is one the principal causes of deviation from Bernoulli's Theorem as derived in § 451 [eq(7)]. The amount of this resistance, often called skin friction, in lbs. (or other unit) for a given extent of rubbing surface is by experiment found

1. To be independent of the pressure between the liquid and solid.
2. To vary nearly with the square of the velocity.
3. To vary directly with the amount of rubbing surface.
4. To vary directly with the heaviness (γ, § 7) of the liquid.

Hence for a given velocity v, a given rubbing surface of area = S, and a liquid of heaviness γ, we may write

$$\text{Amount of friction} = f S \gamma \frac{v^2}{2g} \quad \text{. . . . . . . (1)}$$

in which f is an abstract number called the coefficient of liquid-friction, to be determined by experiment, and for a given liquid, given character (roughness) of surface, and a small range of values for the velocity is approximately constant. The object of introducing the 2 g is not only because $\frac{v^2}{2g}$ is a familiar and use-

ful function of v, but that ($v^2 \div 2g$) is a height or one dimension of length, and $\therefore$ the product of S (surface) by $v^2 \div 2g$ (length) by γ (heaviness) is the weight of an ideal prism of the liquid, and hence is, in quality, one dimension of force. As the friction is also a force, f must be an abstract number and hence the same in all systems of units, in any given case or experiment.

In his experiments at Torquay, England, the late Mr. Froude found the following values for f, the liquid being water, while the rigid surfaces were the two sides of a thin wooden board $3/_{10}$ in. thick and $19^{in.}$ high, coated or prepared in various ways, and drawn edgwise thro' the water at a constant velocity, the resistance being measured by a dynamometer.

TABLE F; Mr. FROUDE'S RESULTS.

[The veloc. was the same = 10 ft. per sec. in each of the following cases. For other velocities the resistance was found to vary nearly as the square of the velocity, the index of the power varying from 1.87 to 2.16]

Character of surface	Value of f			
	When the board was			
	2 ft. long	8 ft. long	20 ft. long	50 ft. long
Varnish, f =	.0042	.0033	.0029	.0026
Paraffine, f =	.0039	.0032	.0028	
Tinfoil, "	.0031	.0029	.0027	.0026
Calico, "	.0089	.0064	.0055	.0048
Fine sand, "	.0083	.0060	.0049	.0041
Medium Sand, "	.0092	.0064	.0055	[illegible]
Coarse Sand, "	.0112	.0073	.0061	

N.B. These numbers diminished by 2 per-cent, and then multiplied by 100 give the mean frictional resistance in lbs. per sq. foot of surface, in each case.

For use in formulae bearing on flow in pipes, f is best determined directly by experiments of that very nature, the results of which will be given as soon as the proper formula has been established.

470. BERNOULLI'S THEOREM FOR STEADY FLOW WITH FRICTION. [The student will now re-read the first part of § 451, as far as eq.(1).]

Considering free any lamina of fluid, Fig. 555, (according to the subdivision of the stream agreed upon in § 451 referred to), the friction on the edges is the only additional force as compared with the system in Fig. 524. Let w denote the length of the wetted perimeter of the base of this lamina; (in case of a pipe running full as we here postulate the wetted perimeter is the whole perimeter, but in the case of an open channel, as a canal, w is only a portion of the whole perimeter of the cross-section). Then

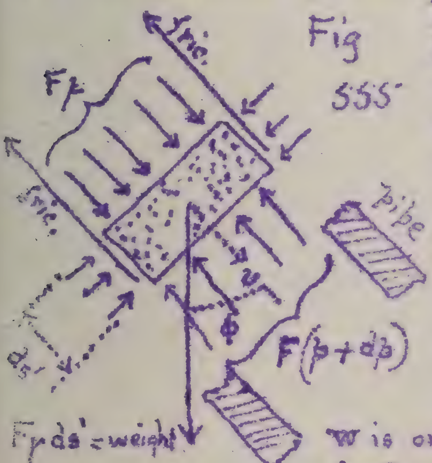


Fig 555

since the rubbing surface of the edge is $S = w\,ds'$, the total friction for the lamina is, by eq.(1) § 469, $f w \gamma\,(v^2 \div 2g)\,ds'$. Hence from $v\,dv = (\text{tan. accel.}) \times ds$ and from Tan. acc. = (Tang. components of acting forces) ÷ mass of lamina, we have

$$v\,dv = \frac{Fp - F(p+dp) + F\gamma\,ds'\cos\phi - f w \gamma \frac{v^2}{2g} ds'}{F\gamma\,ds' \div g} \cdot ds$$

As before, in § 451, considering the simultaneous advance of all the laminae lying between any two sections

m and n, during the small time dt, putting ds' $= ds$, and $ds' \cos\phi = dz$ (see Fig 556) we have, for any one lamina

$$\frac{1}{g}\, v\,dv + \frac{1}{\gamma}\, dp + dz = -f\,\frac{W}{F}\cdot\frac{v^2}{2g}\, ds \quad\ldots(1)$$

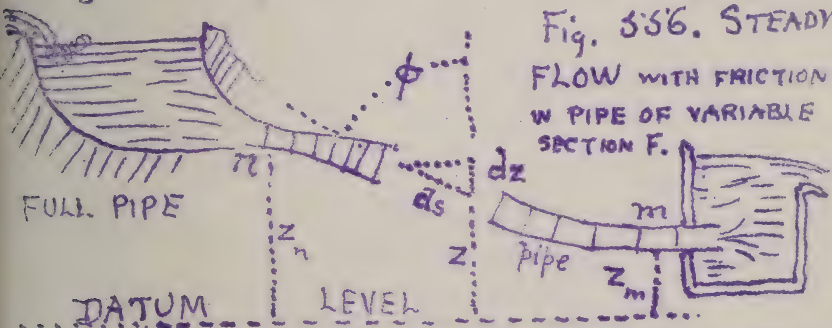


Fig. 556. STEADY FLOW WITH FRICTION IN PIPE OF VARIABLE SECTION F.

Adding up corresponding terms in the infinite number of equations arising from forming one like eq.(1) for each lamina between n and m, for a simultaneous dt, we have, remembering that for a liquid γ is the same in all laminae, (also treating f as the same at all points)

$$\frac{1}{g}\int_n^m v\,dv + \frac{1}{\gamma}\int_n^m dp + \int_n^m dz = -\frac{f}{2g}\int_n^m \frac{W}{F}\, v^2 ds \quad\ldots(2.)$$

i.e. after transposition, and writing, for brevity, $F \div W = R$,

$$\frac{v_m^2}{2g} + \frac{p_m}{\gamma} + z_m = \frac{v_n^2}{2g} + \frac{p_n}{\gamma} + z_n - \frac{f}{2g}\int_n^m \frac{v^2 ds}{R} \quad\ldots\ldots(3.)$$

This is BERNOULLI'S THEOREM for steady flow of a liquid in a pipe of slightly varying sectional area F and (internal) perimeter W, taking into account the "skin friction" alone. Resistances due to the internal friction ~~occasioned by~~ of eddying occasioned by sudden changes of cross section of pipe, elbows, sharp curves, and valve-gates will be mentioned later. The negative term on

The right in (3) is of course a <u>height</u> or <u>head</u> (one dimension of length) as all the other terms are such, and since it is the amount by which the sum of the three heads at m (viz. velocity-head, pressure-head, and potential head), the down-stream locality, lacks of being equal to the sum of the corresponding heads at n, the up-stream locality or section, we may call it

the "LOSS OF HEAD" due to skin-friction between n and m; also called friction-head; or resistance head; or height of resistance.

The quantity $R = F \div w =$ sectional-area $\div$ wetted-perimeter, is an imaginary line or length called the *Hydraulic Mean Radius*; or *Hydraulic Mean Depth*, or simply hydraulic radius of the section. For a circular pipe of diameter $= d$, $R = \frac{1}{4}\pi d^2 \div \pi d = \frac{1}{4} d$ while for a rectangular section of sides $= a$ and b $\Big\} \cdots R = ab \div 2(a+b) = \dfrac{ab}{2(a+b)}$

471. PROBLEMS INVOLVING FRICTION-HEADS, and EXAMPLES OF BER.S THEOREM WITH FRICTION.

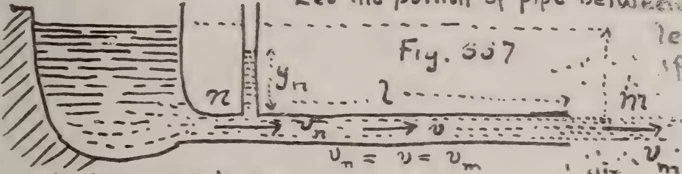


Fig. 557

Let the portion of pipe between n and m be level and of uniform section, circular, of diam. $= d$. The jet at m discharges into the air and has the same sectional area $F = \frac{1}{4}\pi d^2$, as the pipe; Then the press.-head at m is $\frac{p_m}{\gamma} = b$ $= 34$ feet (for water) and the veloc.-head at m is $=$ that at n, since $v_m = v_n$. The height of the water column in the open piezometer at n is noted, and $= y_n$, (so that the press. head at n is $\dfrac{p_n}{\gamma} = y_n + b$) and the length of pipe from n to m is $= l$.

Knowing l, d, y_n and having measured the volume

Q, of flow per time-unit, it is required to find the form of the friction-head and the value of f.

From $F_m v_m = Q$, or $\frac{1}{4}\pi d^2 v_m = Q$ ....(1)

v_m becomes known, and $= v_n$. Also the velocity v is the same for all the ds's of the pipe, since F is constant along the pipe and $Fv = Q$. The hydraulic radius $= \frac{1}{4} d$ (see § 470) ..... (2)

and is the same for all the ds's between n and m.

Substituting now in eq.(3) of § 470, with the axis of the pipe as a datum for potential-heads, we have

$$\frac{v_m^2}{2g} + b + 0 = \frac{v_n^2}{2g} + y_n + z + 0 - \frac{f}{\frac{1}{4}d}\,\frac{v^2}{2g}\int_n^m ds \quad ..(3)$$

i.e. since $\int_n^m ds = l =$ length of pipe from n to m, the friction-head for a pipe of length $= l$, and uniform circular section of diam. $= d$ reduces to the form

$$\text{FRIC. HEAD} = 4f\,\frac{l}{d}\,\frac{v^2}{2g} \;\ldots\ldots\ldots (4)$$

[where v is the velocity of the water in the pipe (in this case $v_m = v_n = v$)] and ∴ varies directly as the length, and the square of the velocity, and inversely as the diameter; also directly as the coefficient f. From (3), then, we derive for this particular problem

$$\text{piez. height} = y_n = 4f\,\frac{l}{d}\,\frac{v^2}{2g} \;\ldots\ldots (5)$$

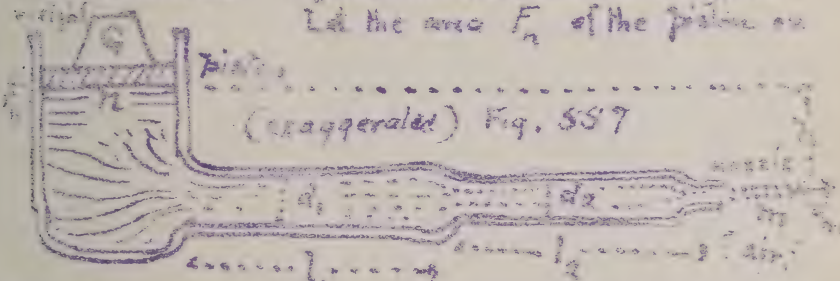
i.e. the open-piezometer height at n is equal to the loss of head (all of which is friction-head here) sustained between n and the mouth of the pipe.

Example. Required the value of f, knowing that $d = 3$ in., $y_n = 10.4$ feet and $Q = 0.1960$ cub.ft. per sec., while $l = 400$ ft (n to m). From eq.(1), we find (with ft.-lb.-sec. system) the velocity in the pipe to be $v = \frac{4Q}{\pi d^2} = \frac{4 \times 0.196}{\pi\left(\frac{1}{4}\right)^2} = 4.0$ ft. per sec.

then using eq.(5) we determine f to be

$$f = \frac{2 g d y_n}{4 l v^2} = \frac{2 \times 32.2 \times \frac{1}{4} \times 10.7}{4 \times 400 \times 4^2} = 0.0065$$

PROBLEM II. *Hydraulic Accumulator.* Fig. 557. Let the area F_n of the piston on



(exaggerated) Fig. 557

the left be quite large compared with that of the section of the nozzle on the right, F_m. The cylinder contains a frictionless weighted piston producing (so long as its downward slow motion is uniform) a fluid pressure on its lower face of an intensity (per unit area) $p_n = \frac{G + F_n p_a}{F_n}$ where p_a = one atmos. Hence the press. head at n is ....

$$\frac{p_n}{\gamma} = \frac{G}{F_n \gamma} + b \quad \text{------} \quad (6)$$

G = weight of load + piston

The jet is of section F_m = that of the small straight nozzle (no contraction). The junctions of the two pipes with each other, and with the cylinder and short nozzle are all smoothly rounded; hence the only losses of head in steady flow between n and m are the friction-heads in the two long pipes, neglecting that in the short nozzle. These friction-heads will be of the form in eq.(4) and will involve the velocities v_1 and v_2 in these pipes, supposed running full. Knowing G, and all dimensions and heights, it is required to find the velocity v_m of the jet, flowing into the air, and the volume Q of flow per time-unit, assuming f to be known and to be the same in both pipes, (not strictly true in fact).

Let the lengths and diameters of the pipes be as in Fig. 557, their sectional areas F_1 and F_2, and the unknown velocities in them v_1 and v_2. From the equation of continuity [eq. (3) § 449] we have

$$v_1 = \frac{F_m v_m}{F_1} \text{ and } v_2 = \frac{F_m v_m}{F_2} \quad \ldots\ldots\ldots (7)$$

To find v_m we apply B's theorem with friction (eq. 3 § 470), taking the down-stream position m in the jet close to the nozzle, and the up-stream position n just under the piston in the cylinder where the velocity v_n is practically nothing. Hence, with m as datum plane

$$\frac{v_m^2}{2g} + 0 + 0 = 0 + \frac{p_n}{\gamma} + h - 4f\frac{l_1}{d_1}\frac{v_1^2}{2g} - 4f\frac{l_2}{d_2}\frac{v_2^2}{2g} \ldots (8)$$

Apparently (8) contains three unknown quantities, v_m, v_1, and v_2; but from eqs. (7) v_1 and v_2 can be expressed in terms of v_m, whence ~~[illegible]~~ (see also eq. 6)

$$\frac{v_m^2}{2g}\left[1 + 4f\frac{l_1}{d_1}\left(\frac{F_m}{F_1}\right)^2 + 4f\frac{l_2}{d_2}\left(\frac{F_m}{F_2}\right)^2\right] = h + \frac{G}{F_n \gamma} \ldots (9)$$

and finally

$$v_m = \frac{\sqrt{2g\left[h + \frac{G}{F_n\gamma}\right]}}{\sqrt{1 + 4f\frac{l_1}{d_1}\left(\frac{F_m}{F_1}\right)^2 + 4f\frac{l_2}{d_2}\left(\frac{F_m}{F_2}\right)^2}} \quad \ldots\ldots (10)$$

and

$$Q = F_m v_m \ldots (11)$$

Example. If we replace the force G in this problem by the thrust P exerted along the pump piston of a steam fire engine, we may treat the foregoing as a close approximation to the practical problem of such an apparatus, the pipes being consecutive straight lengths of hose in which for the probable values of v_1 and v_2 we may take $f = .0075$ (See "Fire Streams"

by Geo. Ellis, Springfield, Mass.) (Strictly f depends, to a certain extent upon the velocity.) Let $P = 12000$ lbs., and the piston area at $n = F_n = 72$ sq. in. $= \frac{1}{2}$ sq. ft. Let $h = 20$ ft., and the dimensions of the hose as follows: $d_1 = 3$ in., $d_2 = 2$ in., d_m (of nozzle) $= 1^{in}$, $l_1 = 400$ ft., $l_2 = 500$ ft. With the foot-pound-sec. system, eq. 10 gives

$$v_m = \sqrt{\frac{2 \times 32.2\left(20 + \dfrac{12000}{\frac{1}{2} \times 62.5}\right)}{1 + 4 \times .0075\left[\dfrac{400}{\frac{1}{4}}\left(\dfrac{1}{9}\right)^2 + \dfrac{500}{\frac{1}{6}}\left(\dfrac{1}{4}\right)^2\right]}}$$

$$= \sqrt{\frac{2 \times 32.2 \times 414}{1 + 0.59 + 5.62}} = 76.53 \text{ ft. per sec.}$$

If this jet were directed vertically upward it should theoretically reach a height $= v_m^2 \div 2g$ = nearly 90 feet, but the resistance of the air would reduce this to about 55 feet.

Further, eq. (11), $Q = F_m v_m = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 76.53 = \left\{\begin{matrix} 4.18 \\ \text{cub. ft.} \\ \text{per} \\ \text{sec.} \end{matrix}\right.$

If there were no resistance in the hose we should have $v_m = \sqrt{2g\left[\frac{p}{F_n \gamma} + h\right]}$, see § 534, $= \sqrt{2g\, 414^{ft}} = 191.5 \left(\begin{matrix}\text{ft} \\ \text{per} \\ \text{sec.}\end{matrix}\right.$

472. LOSS OF HEAD IN ORIFICES AND SHORT PIPES. So long as the steady flow between two localities n and m takes place in a pipe having no abrupt enlargement or diminution of section nor sharp curves, bends, or elbows, the loss of head is ascribed solely to surface-(or skin-) friction, but the introduction of any of the above mentioned features occasions eddying and consequent internal friction and heat, thereby causing additional deviations from Bernoulli's theorem, i.e., ad-

ditional losses of head or heights of resistance. From the analogy of the form of a friction-head in a long pipe (eq. 4 § 471) we may assume that any of the above heights of resistance is proportional to the square of the velocity, and may ∴ always be written in the form

$$\left.\begin{array}{l}\text{LOSS OF HEAD, due to}\\ \text{any cause except skin friction}\end{array}\right\} = \zeta \frac{v^2}{2g} \quad \ldots\ldots\ldots (1)$$

in which v is the velocity of the water in the pipe at the section where the resistance occurs, or, if on account of an abrupt enlargement of the stream-section there is a corresponding diminution of velocity, then v is always to denote this diminished velocity, in the down-stream section. This velocity v is often an unknown quantity at the outset. ζ, corresponding to the abstract factor $4f\frac{l}{d}$ in the height of resistance due to skin friction (eq. 4 § 471) is an abstract number called the CO-EFFICIENT OF RESISTANCE to be determined by experiment, or computed theoretically where possible. It is, in the main, independent of the velocity for a given fitting, casing, pipe joint, valve-gate in definite position, etc., etc.

473. HEIGHTS OF RESISTANCE (OR LOSSES OF HEAD) OCCASIONED BY SHORT CYLINDRICAL TUBES. When dealing with short tubes discharging into the air in § 466, deviations from Bernoulli's theorem were made good by using a co-efficient of velocity ϕ, dependent on experiment. This device answered every purpose for the simple circumstances of the case, as well as for simple orifices. But the great variety of possible designs of compound pipes (with skin-friction, bends, sudden changes of section, etc.) renders it almost impossible, in such pipes, to provide for deviations from B's theorem by a single co-efficient of velocity, as new experiments would be needed for each new design of

pipe. Hence the great utility of the conception "loss of head", one for each source of resistance.

If a long pipe issues from the plane side of a reservoir and the corners of the junction are <u>not</u> <u>rounded</u>, (see Fig. 559) we shall need an expression for the loss of head at entrance, E, as well as that $\left(= 4f\,\frac{l}{d}\cdot\frac{v^2}{2g}\right)$ due to the skin friction in the pipe.

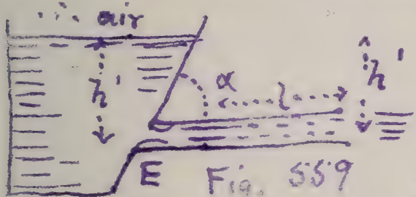


Fig. 559

But whatever the velocity v, in the pipe, is going to be, influenced as it is both by the entrance loss of head and the skin-friction-head (in applying B's theorem), the loss of head at E viz. $\zeta\,\frac{v^2}{2g}$, will be just the same as if efflux at E took place through enough of the pipe to constitute a short pipe discharging into the air under some head h (different from h' of Fig. 559) sufficient to produce the same velocity v. But in that case we should have

$$v = \phi\sqrt{2gh}\ , \text{ or } \frac{v^2}{2g} = \phi^2 h \;\ldots\ldots\; (1)$$

(See §§ 466 and 467, ϕ being the "co-efficient of velocity" and h the head in the cases mentioned in in those articles)

We therefore apply B's theorem to the cases of those articles, see Figs. 548 and 552, in order to determine the loss of head due to the short pipe, and [illegible] (with m as datum-level for potential heads)

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h - \zeta_E\,\frac{v^2}{2g} \;\ldots\ldots\; (2)$$

Now the v of eq. (2) is = the v_m of the Figs referr-

in to, and ζ_E is a coefficient of resistance, for the short pipe, whose value we seek. Substituting for $\frac{v^2}{2g}\left(=\frac{v_m^2}{2g}\right)$ its value $\phi^2 h$ from eq. (1), we have

$$\zeta_E = \frac{1}{\phi^2} - 1 \quad \text{.............} \quad (3.)$$

Hence when $\alpha = 90°$ (i.e. the pipe is ⊥ to the inner reservoir surface), we derive

$$\zeta_E = \frac{1}{\phi_0^2} - 1 = \frac{1}{(0.815)^2} - 1 = 0.505 \quad .(4) \quad \begin{cases}\text{for} \\ \alpha = 90°\end{cases}$$

and similarly for other values of α (taking ϕ from the table § 467) we compute the following values of ζ_E (abstract number) for the loss of head $\zeta_E \frac{v^2}{2g}$ at the entrance of a pipe, corners not rounded; see Fig. 559

For α =	90°	80°	70°	60°	50°	40°	30°
ζ_E =	.505	.565	.635	.713	.794	.870	.937

From eq. (4) we see that the loss of head at the entrance of the pipe, corners not rounded, with $\alpha = 90°$ is about one half, $\left(\frac{505}{1000}\right)$ of the height due to the velocity v in that part of the pipe; (v = same all along the pipe if cylindrical.) The value of v, Fig 559, itself, depends on all the features of the design from reservoir to nozzle. See § 475.

If the corners at E are properly rounded, the entrance loss of head may practically be done away with; still if v is quite small (as it may frequently be, from large losses of head further down stream) the saving thus secured, while helping to increase v slightly (and thus the saving itself) is insignificant.

474. GENERAL FORM OF BERNOULLI'S THEOREM CONSIDERING ALL LOSSES OF HEAD.

In view of preceding explanations and assump–

tions, we may write, in a final and general form, Bernoulli's Theorem for a steady flow from an up-stream locality n to a down-stream locality m, viz.:

$$\frac{v_m^2}{2g} + \frac{p_m}{\gamma} + z_m = \frac{v_n^2}{2g} + \frac{p_n}{\gamma} + z_n - \left[\begin{array}{l}\text{All losses of head}\\ \text{occurring between}\\ n \text{ and } m\end{array}\right] \ldots$$

Each loss of head, (or height of resistance) will be of the form $\zeta \frac{v^2}{2g}$, except skin-friction head for long pipes viz. $4 f \frac{l}{d} \frac{v^2}{2g}$, the v in each case being the velocity, known or unknown, in that part of the pipe where the resistance occurs, (and hence is not necessarily equal to v_m or v_n.)

474a. THE CO-EFFICIENT f, FOR SKIN-FRICTION OF WATER IN PIPES. See eq.(1) § 469. Experiments have been made by Weisbach, Eytelwein, Darcy, Bossut, Prony, Du Buat, Fanning, etc., to determine f in cylindrical pipes of various materials (tin, glass, zinc, lead, brass and iron) of diameters from $\frac{1}{2}$ inch up to 36 inches. In general it may be stated that these experiments prove the following:

LAW 1. That f decreases when the velocity increases
E.g. In one case with the same pipe f was $= .0070$ for $v = 2$ feet per sec. and " $= .0056$ " $v = 20$

LAW 2. That f decreases slightly as the diameter increases.
E.g. In one case, for same veloc., f was $= .0069$ in a 3 in. pipe while f " $= .0064$ in a 6 in. pipe

LAW 3. That (as in the case of sliding friction between solids) the condition of the inner surface of the pipe affects the value of f.
E.g. Darcy found, with a foul iron pipe, $d = 10$ in., vel. $= 3.67$ ft. per sec. and $f = .0113$; whereas Fanning (see p. 238 of his "Water Supply Engineering") with a cement-lined pipe, found $f = .0052$ when the velocity was 3.74 ft. per sec., d being $= 20$ inches.

Weisbach, finding Law 1 very prominent proposed the formula

$$f = 0.00359 + \frac{0.00429}{\sqrt{v \text{ (in ft. per sec.)}}} \quad \left\{\begin{array}{l}\text{when the velocities}\\ \text{were considerable;}\end{array}\right.$$

while Darcy, taking into account both Law 1 and Law 2, puts .... (see p. 585 Rankine's Applied Mechanics)

(pass to p. 147)

TABLE OF THE CO-EFFICIENT, f, FOR FRICTION IN CLEAN IRON PIPES; for WATER.

Vel. in ft. per sec.	diam. $= \frac{1}{2}$ in. $= .0417$ ft.	diam. $= 1$ in. $.0834$	2 in. .1667	3 in. .25	4 in. .333 ft.	6 in. .50 ft.	8 in. .667 ft.	10 in. .833	12 in. 1.0 ft.	16 in. 1.333	20 in. 1.667	30 in. 2.5	40 in. 3.333	60 in. 5.0 ft.
0.1	.0150	.0119	.00870	.00800	.00763	.00730	.00704	.00684	.00667	.00623				
0.3	.0137	.0113	856	784	750	720	693	673	657	614	.00578			
0.6	.0124	.0104	822	767	732	702	677	659	642	608	567	.00504	.00434	.00357
1.0	.0110	.00950	790	743	712	684	657	643	624	586	555	492	428	353
1.5	.00959	868	757	720	693	662	640	625	607	572	542	482	421	349
2.0	862	810	731	700	678	656	624	607	593	557	529	470	416	346
2.5	795	768	710	683	662	634	611	596	581	548	518	460	410	342
3.0	.00753	.00734	.00692	.00670	.00650	.00623	.00600	.00584	.00570	.00538	.00509	.00452	.00401	.00337
4.0	722	702	671	651	631	607	586	568	553	524	498	441	400	333
6.0	.00689	670	640	622	605	582	562	548	534	507	482	430	391	324
8.0	663	646	618	600	587	562	544	532	520	491	470	422	384	320
12.	630	607	590	582	560	540	522	512	500	478	457	412	377	.00313
16.	.00618	.00600	.00581	.00570	.00552	.00530	.00519	.00502	.00491	.00470	.00450	.00406	.00370	[illegible]
20.	615	598	579	566	549	525	508	498	485					

$$f = .0043\left(1 + \frac{1}{12 \times \text{diam in ft}}\right) + \frac{.001}{v^{\text{ft per sec}}}\left(1 + \frac{1}{18 \times \text{diam in ft.}}\right)$$

For practical purposes, Mr. J. T. Fanning has arranged in an extensive table (pp. 242 to 246 of the book referred to above) values of f for clean iron pipe, of diams from $\frac{1}{2}$ in. to 90 in., and for velocities of 0.1 ft. to 20 ft. per sec. Of this the table on the preceding page is an abridgement, and will be used in working numerical problems.

In obtaining f for slightly tuberculated and for foul pipes, the recommendations of Mr. Fanning seem to justify the following:

For slightly tuber^{ed} pipes of diams. =	$\frac{1}{2}$ ft	1 ft	2 ft	4 ft.
we should add	23 %	34 %	16 %	13 %;
while for foul pipes of same size we should add	72 %	60 %	38 %	25 %

of the f of Fannings table for clean pipes, to itself.

Example. If $f = .007$ for a certain $\frac{1}{2}$ ft. pipe when clean, we have $f = .007 \times 1.72 = .01204$ when it is foul.

For first approximations a mean value of $f = .006$ may be employed, since in some problems sufficient data may not be known in advance to enable us to find f from the table.

Example. Fig. 560. In the steady pumping of crude petroleum weighing $\gamma = 55$ lbs. per cubic foot, thro' a six inch pipe 30 miles long to a station 700 ft. higher than the pump, it is found that the pressure in the pump cylinder at n, necessary to keep up a velocity of 4.4 ft. per second in the pipe was 1000 lbs. per sq. inch. Required the co-efficient f in the pipe. As all losses except the friction-head in the pipe are insignificant, the latter only will be considered. The velocity-head at n may be put = 0; the jet at m being of the same size as the pipe, the velocity in the pipe is $= v_m$ and $\therefore v_m = 4.4$ ft. per sec. Notice that m, the down-stream section, is at a higher level than n.

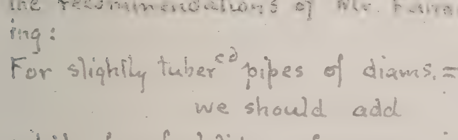


Fig. 560

From B's Theorem, § 474, we have, with n as a

datum level $\frac{v_m^2}{2g} + b + h = 0 + \frac{p_n}{\gamma} + 0 - 4f\frac{l}{d}\frac{v^2}{2g}$ ....(1)

Using the ft., lb., and sec., we have $h = 700$ ft., $\frac{v_m^2}{2g} = 0.30$ ft. while $b = \frac{14.7 \times 144}{55} = 38.47$ ft., & $\frac{p_n}{\gamma} = \frac{1000 \times 144}{55} = 2880$ ft.
∴, in eq. (1),

$$0.30 + 38.5 + 700 = 2880 - 4f\,\frac{30 \times 5280}{\frac{1}{2}}\,\frac{(4.4)^2}{64.4} \quad \ldots\ldots(2)$$

Solving, we derive $f = .00560$; (whereas for water, with $v = 4.4'$ per sec. and $d = \frac{1}{2}$ ft., the table, p. 146, gives $f = .00[illegible]$)

475. FLOW THRO' A LONG STRAIGHT CYLINDRICAL PIPE, including both friction-head and entrance loss of head, (corners not rounded). Reservoir large. Fig. 560. The jet issues directly from the end of the pipe, in ∥ filaments, into the air, and ∴ has the same section; hence, also, v_m of the jet $= v$ in the pipe, which is

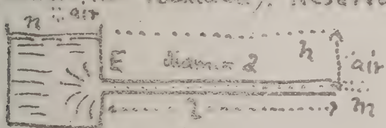


Fig. 560

assumed to be running full, and is ∴ the velocity to be used in the loss of head $\zeta_E \frac{v^2}{2g}$ at the entrance E (§ 473).

Taking m and n as in figure and applying B's theorem (§ 474) with m as datum level for potential heads z_m and z_n we have

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h - \zeta_E\frac{v^2}{2g} - 4f\frac{l}{d}\frac{v^2}{2g} \quad \ldots\ldots(1)$$

Three different problems may now be solved:

First, required the head h to keep up a flow of given vol. $= Q$ per unit of time in a pipe of given length l and diam. d. From the eq. of continuity we have $Q = F_m v_m = \frac{\pi}{4} d^2 v_m$

$$\therefore \quad v_m = \text{vel. of jet} = \text{vel. in pipe} = \frac{4Q}{\pi d^2} \quad \ldots\ldots(2)$$

Having found $v_m = v$ from (2), we obtain from (1) the required

$$h = \frac{v^2}{2g}\left[1 + \zeta_E + 4f\frac{l}{d}\right] \ldots \text{becomes known} \ldots(3)$$

$\zeta_E = .505$ if $\alpha = 90°$ (see § 473), while f may be taken from the table, § 474 a, p. 146, for the given diam. and computed velocity, if the pipe is <u>clean</u>; if not clean see mid. p. 147, for slightly tuberculated and foul pipes.

Secondly, Given the head h, and the length l and diam. d of pipe, required the velocity in the pipe $v = v_m$ = that of jet, and the volume delivered per unit of time Q. Solving (1) for v_m we have

$$v_m = \sqrt{\frac{1}{1+\zeta_E+4f\frac{l}{d}}}\sqrt{2gh} \quad \ldots\ldots\ldots(4)$$

$$\text{whence } Q = \tfrac{1}{4}\pi d^2 v_m \ldots(5)$$

The first radical in eq.(4) may for brevity be called a coefficient of velocity, ϕ, for this case. Since the jet has the same diameter as the pipe this radical may also be called a co-efficient of efflux. Since in (4) f depends on the unknown v as well as the known d we must first put $f = .006$ for a first approximation for v_m; then take a corresponding value for f and subst. again

Thirdly, knowing the length of pipe and the head h, we wish to find the proper diameter d for the pipe to deliver a given volume Q of water per unit of time Now $v = v_m = \frac{Q}{\frac{1}{4}\pi d^2}$ .....(6)
which substituted in (1) gives

$$2gh = \left(\frac{4Q}{\pi}\right)^2\frac{1}{d^4}\left[1+\zeta_E+4f\frac{l}{d}\right] = \left(\frac{4Q}{\pi}\right)^2\left[\frac{1+\zeta_E}{d^4}+\frac{4fl}{d^5}\right]; \text{ i.e}$$

$$2ghd^5 = \left(\frac{4Q}{\pi}\right)^2\left[(1+\zeta_E)d+4fl\right] \therefore d = \sqrt[5]{\frac{(1+\zeta_E)d+4fl}{2gh}\left(\frac{4Q}{\pi}\right)^2} \ldots(7)$$

in which $\zeta_E = .505$ for $\alpha = 90°$ (§ 472)

As the radical contains d, we first assume a value for d, with $f = .006$ and substitute in (7). With the approx. value of d thus obtained, substitute again with a new value for f based on the value of v in (6) obtained from the first approx. value of d. And thus a still closer value for d is derived; and so on. If l is quite large we may put $d = 0$ for the first approximation. See last figure for these examples

Example 1. What head h is necessary to deliver 120 cub. ft. of water per minute through a clean straight iron pipe 140 ft. long and 6 inches in diameter. From eq. (2) (ft. lb. sec.) we have

$$v = v_m = \frac{4 \times \frac{120}{60}}{\pi\left(\frac{1}{2}\right)^2} = 10.18 \text{ ft. per sec.}$$

Now for $v = 10'$ per sec. and $d = \frac{1}{2}$ ft. = 6 in., we find in the table of § 474, $f = .00549$, and $\therefore$ from eq. (3) we have

$$h = \frac{(10.18)^2}{2 \times 32.2}\left[1 + .505 + \frac{4 \times .00549 \times 140}{\frac{1}{2}}\right] = 12.23 \text{ feet.}$$

of which total head (as we may call it) 1.60 feet is used in producing the velocity 10.18 (i.e. $v_m^2 \div 2g = 1.60$ ft.), while 0.808 ft. ($= \zeta_E \frac{v_m^2}{2g}$) is lost at the entrance E ($\alpha = 90°$) and 10.82 ft. (friction-head) is lost in skin-friction.

Example 2. (Data from Weisbach) Required the delivery, Q, thro' a straight clean iron pipe 48 ft. long and 2 in. in diam, with 5 ft. head ($=h$). $v=v_m$ being unknown we take $f=.006$ for a first approximation and obtain, from eq. (4), ($\alpha=90°$)

$$v_m=\sqrt{\frac{1}{1+.505+\frac{4\times.006\times48}{1/6}}}\sqrt{2\times32.2\times5}=6.18 \text{ ft. per sec}$$

From the table §474, for $v=6.2$ ft. per sec. and $d=2$ in. we find $f=.00638$, whence we now have more accurately,

$$v_m=\sqrt{\frac{1}{1+.505+\frac{4\times.00638\times48}{1/6}}}\sqrt{2\times32.2\times5}=6.04 \text{ ft. p. sec.}$$

which is sufficiently close. Then the volume delivered per sec. is $Q=\frac{1}{4}\pi d^2 v_m=\frac{1}{4}\pi\left(\frac{1}{6}\right)^2 6.04=0.1307$ cub. ft. p. sec.

Example 3. (Data from Weisbach) What must be the diameter of a straight clean iron pipe 100 ft. in length, which is to deliver $Q=\frac{1}{2}$ of a cubic foot of water per sec. under 5 ft. head ($=h$)?

With f approx. $=.006$ we have from eq. 7, putting $d=0$ under the radical for a first trial, (ft. lb. sec., using logarithms)

$$d=\sqrt[5]{\frac{4\times.006\times100}{2\times32.2\times5}\cdot\left(\frac{4\cdot\frac{1}{2}}{\pi}\right)^2}=\text{about } 0.30 \text{ ft.}$$

whence a rough approx. for $v=v_m$ is

$$v=\frac{4Q}{\pi d^2}=\frac{4\times\frac{1}{2}}{\pi\times.09}=7 \text{ ft. per sec.}$$

Hence with $d=0.30$ ft which $=3.6$ in., and $v=7$, $f=.00601$

$$\therefore\ d=\sqrt[5]{\frac{1.505\times.30+4\times.00601\times100}{2\times32.2\times5}\left(\frac{4\times\frac{1}{2}}{\pi}\right)^2}=0.324 \text{ ft.}$$

with $d=0.324$ ft $v=\frac{4\times\frac{1}{2}}{\pi(.324)^2}=6.06$ ft. per sec, we have finally with $f=.00609$

$$d=\sqrt[5]{\frac{1.505\times0.325+4\times.00609\times100}{2\times32.2\times5}\cdot\left(\frac{2}{\pi}\right)^2}=0.326 \text{ ft.}$$

475 a. CHÉZY'S FORMULA. If in the problem of the preceding paragraph so long, and $\therefore$ $l:d$ so great, that $4fl\div d$ in eq.(3) is very large compared with $1+\zeta_E$, we may neglect the latter term, whence eq.(3) reduces to

$$h=4f\frac{l}{d}\cdot\frac{v_m^2}{2g} \quad \text{.....(pipe very long; see Fig. 560).....} \quad (8)$$

which is known as *Chézy's formula*. *For Example*,

if $l = 100$ ft and $d = 2$ in. $= \frac{1}{6}$ ft. and f approx. $= .006$, we should have $4f\frac{l}{d} = 144$, while $1+\zeta_E$ for square corners $= 1.505$ only.

If in (8) we substitute $v_m = \frac{Q}{F_m} = Q \div \frac{1}{4}\pi d^2$, 8 reduces to

$$h = \frac{16}{\pi^2} \cdot \frac{fl}{d^5} \cdot \frac{Q^2}{2g} \text{ .... (very long pipe) .... (9)}$$

so that for a very long pipe, considering f as approx.^{ly} constant, we may say that to deliver a vol $= Q$ per unit of time thro' a pipe of given length $= l$, the necessary head, h, is inversely proportional to the fifth power of the diameter. Eq. (9) may be stated in still other forms.

476. COEFFICIENT f IN FIRE-ENGINE HOSE. Mr. Geo. A. Ellis, C.E., in his little book on "Fire-Streams", describing experiments made in Springfield, Mass., gives a graphic comparison (p. 45 of his book) of the friction-heads occurring in rubber hose, in leather hose, and in **clean** iron pipe, each of $2\frac{1}{2}$ in. diameter with various velocities; on which the following statements may be based: That for the given size of hose and pipe ($d = 2\frac{1}{2}$ in.) the co-efficient f for the leather and rubber hose respectively may be obtained approximately by adding to f for clean iron pipe (and a given velocity) the per cent. of itself shown in the accompanying table; *Example.* For a clean iron pipe $2\frac{1}{2}$ in. diam., for a veloc. $= 10$ ft. per sec. we have from § 474 a, $f = .00593$. Hence for a leather hose of the same diam., we have for $v = 10'$ per sec.,

$$f = .00593 + .43 \times .00593$$
$$= .00848$$

Velocity ft. per sec.	Rubber hose $2\frac{1}{2}$ in. diam.	Leather hose $2\frac{1}{2}$ in. diam.
3.0	50 %	300 %
6.5	20	80.
10.	16	43.
13.	12.5	32.
16.	12.	30.

477. BERNOULLI'S THEOREM AS AN EXPRESSION OF THE CONSERVATION OF ENERGY FOR THE LIQUID PARTICLES. In any kind of flow *without friction*, steady or not, in rigid immovable vessels the aggregate potential and kinetic energy of the whole mass of liquid concerned is necessarily a constant quantity (see §§ 148 and 149) but *individual particles* (as the particles in the sinking free surface of water in a vessel which is rapidly being

emptied) may be continually losing potential energy, i.e. reaching lower and lower levels, without any compensating increase of kinetic energy or of any other kind; but in a *steady* flow without friction in rigid vessels we may state that the stock of energy of a given particle, or small collection of particles, is *constant* during the flow, provided we recognize a third kind of energy which may be called PRESSURE-ENERGY, or capacity for doing work by virtue of internal fluid pressure; as may be thus explained:

In Fig. 561 let water, with a very slow motion and under a pressure p (due to the reservoir-head + atmosphere-head behind it) be admitted behind a piston the space beyond which is *vacuous*. Let s = length of stroke, and " F = the area of piston.

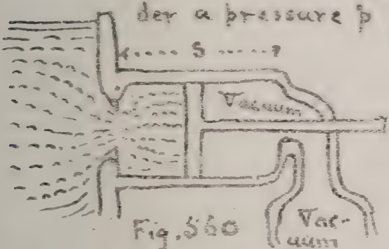


Fig. 560

At the end of the stroke, by motion of proper values, communication with the reservoir is cut off on the left of the piston and opened on the right, while the water in the cylinder now on the left of the piston is put in communication with the vacuous exhaust chamber. As a consequence the internal pressure of this water falls to zero (height of cylinder small) and on the return stroke is simply conveyed out of the cylinder, neither helping nor hindering the motion. That is, in doing the work of one stroke viz. W = force × distance = $Fp \times s = Fps$ a volume of water $V = Fs$, weighing $Fs\gamma$ (lbs. or other unit) has been used, and, in passing thro' the motor, has experienced no appreciable change in velocity (motion slow) and ∴ no change in kinetic energy, nor any change of level, and ∴ " " " potential " ", but it has given up all its pressure.

Now W, the work obtained by the consumption of a weight $= G = V\gamma$ of water, may be written

$$W = Fps = Fsp = Vp = V\gamma \frac{p}{\gamma} = G\frac{p}{\gamma} \quad \ldots\ldots (1)$$

Hence a weight of water $= G$, is capable of doing

the work $G \times \frac{p}{\gamma} = G \times$ head due to pressure p, i.e., $= G \times$ pressure-head, in giving up all its pressure p; or otherwise, while still having a pressure p, a weight G of water possesses an amount of energy, which we may call <u>pressure-energy</u>, of an amount $= G \cdot \frac{p}{\gamma}$, where $\gamma =$ the heaviness (§ 7) of water, and $p/\gamma =$ a height, or head, measuring the pressure p, i.e. it = the pressure-head.

We may now state Bernoulli's Theorem without friction in a new form, viz.: Multiply each term of eq. 7, § 451, by $Q\gamma$ the weight of water flowing per second (or other time unit) in the steady flow, and we have

$$Q\gamma \frac{v_m^2}{2g} + Q\gamma \frac{p_m}{\gamma} + Q\gamma z_m = Q\gamma \frac{v_n^2}{2g} + Q\gamma \frac{p_n}{\gamma} + Q\gamma z_n \ldots\ldots (2)$$

But $Q\gamma \frac{v_m^2}{2g} = \frac{1}{2} \frac{Q\gamma}{g} v_m^2 = \frac{1}{2} \times$ <u>mass</u> flowing per time-unit $\times$ square of the velocity = the Kinetic energy possessed by the vol. Q of water on passing the section m, due to the velocity at m. Also $Q\gamma \frac{p_m}{\gamma} =$ the pressure-energy of the vol. Q at m, due to the pressure at m; while $Q\gamma z_m$ = the potential energy of the vol. Q at m due to its height z_m above the arbitrary datum-plane. Corresponding statements may be made for the terms on the right-hand side of (2) referring to the other section, n, of the pipe. Hence (2) may be thus read: *The aggregate amount of energy (of the three kinds mentioned) resident in the particles of liquid when passing section m is equal to that when passing any other section as n; in steady flow without friction.* That is the store of energy is constant.

478. BERNOULLI'S THEOREM *WITH* FRICTION, FROM THE STANDPOINT OF ENERGY. Multiply each term in the equation of § 474 by $Q\gamma$, as before, and denote a loss of head or height of resistance due to any cause by h_r, and we have

$$Q\gamma \frac{v_m^2}{2g} + Q\gamma \frac{p_m}{\gamma} + Q\gamma z_m = Q\gamma \frac{v_n^2}{2g} + Q\gamma \frac{p_n}{\gamma} + Q\gamma z_n - \sum_n^m Q\gamma h_r \ldots (3)$$

Each term $Q\gamma h_r$ (e.g. $Q\gamma \, 4f \frac{l}{d} \frac{v^2}{2g}$ due to skin-friction in a long pipe; and $Q\gamma \, \zeta_s \frac{v^2}{2g}$ due to loss of head at the re-

servoir entrance of a pipe) represents a loss of energy, occurring between any locality n and any other locality m downstream from n, but is really still in existence in the form of heat generated by the friction of the liquid particles against each other or the sides of the pipes.

As illustrative of several points in this connection, consider

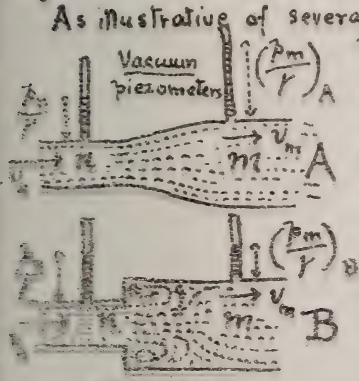


Fig. 562.

two short lengths of pipe in Fig. 562, A and B, one offering a gradual, the other a sudden, enlargement of section, but otherwise identical in dimensions. We suppose them to occupy places in separate lines of pipe in each of which a ~~steady~~-flow with full cross-sections is proceeding and so regulated that the velocity and internal pressure at n, in A, are equal respectively to those at n in B. Hence, if vacuum piezometers be inserted at n, the smaller section, the water columns maintained in them by the internal pressure will be of the same height, $\frac{p_n}{\gamma}$, for both A and B. Since at m, the larger section, the sectional area is the same for both A and B, and since F_n in A $= F_n$ in B so that $Q_A = Q_B$; hence v_m in A $=$ v_m in B and is less than v_n.

How in B a loss of head occurs (and hence a loss of energy) between n and m, but none in A (except slight friction-head), hence in A we should find as much energy present at m as at n, only differently distributed among the three kinds, while at m in B the aggregate energy is $<$ that at n in B.

As regards kinetic energy, there has been a loss between n and m in both A and B, for v_m is $< v_n$, (and equal losses). As to potential energy there is no change between n and m either in A or B, since n and m are on a level. Hence if the loss of kinetic energy in B is not compensated for by an equal gain of pressure-energy, (as it is in A), the pressure-head $\left(\frac{p_m}{\gamma}\right)_B$ at m in B should be less than that $\left(\frac{p_m}{\gamma}\right)_A$ at m in A. Experiment shows this to be true, the loss of

being due to the internal friction in the eddy occasioned by the sudden enlargement; the water column at m in B is found to be of a less height than that at m in A, whereas at n they are equal.

In brief, in A the loss of kinetic energy has been made up in pressure-energy, with no change of potential energy, but in B there is an actual absolute loss of energy $= Q\gamma h_r$, or $= Q\gamma \zeta \frac{v_m^2}{2g}$ suffered by the vol. Q of liquid. The value of ζ in this case and others will be considered in subsequent §§.

Similarly, losses of head, and ∴ losses of energy, occur at elbows, sharp bends, and obstructions causing eddies and internal friction, the amount of each loss **for** a given weight, G, of water, being $= Gh_r = G\zeta\frac{v^2}{2g}$. $h_r = \zeta\frac{v^2}{2g}$ being the loss of head occasioned by the obstruction (§ 474). It is ∴ very important in transmitting water through pipes for purposes of power to use all possible means of preventing disturbance and eddying among the liquid particles: E.g., sharp corners, turns, elbows, abrupt changes of section, should be avoided in the design of the conduit pipe. The amount of the losses of head, or heights of resistance, due to these various causes will now be considered (except skin-friction, already treated) Each such loss of head will be written in the form $\zeta\frac{v^2}{2g}$ and we are principally concerned with the value of the abstract number ζ, or co-efficient of resistance, in each case. The velocity v is the velocity, known or unknown, <u>where the resistance occurs</u> or, if the section of pipe changes at this place, then v = velocity on the <u>down-stream section</u>. Weisbach, of the mining-school of Freiberg, Saxony, has been one of the most noted experimenters in this respect, and will be frequently quoted.

479. LOSS OF HEAD DUE TO SUDDEN (i.e. SQUARE-EDGED) ENLARGEMENT. BORDA'S FORMULA. Fig 563

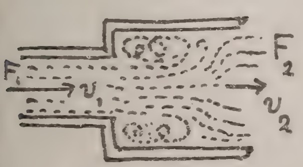


Fig. 563.

An eddy is formed in the angle with consequent loss of energy. Since each particle of water of weight = G, arriving with the veloc. v_1 in the small pipe, may be considered to have an *impact* against the base of the infinitely great and more slowly moving column of water in the large pipe,

and after the impact moves on with the same velocity, v_2, as that column, just as occurs in *inelastic direct central impact* (§ 60), we may find the energy lost by this particle on account of this impact by eq. (1) of § 138, in which putting $M_1 = G_1 \div g$, and $M_2 = G_2 \div g$ = mass of infinitely great body of water in the large pipe, so that $M_2 = \infty$ we have

$$\text{Energy lost by particle} = G_1 \frac{(v_1 - v_2)^2}{2g} \quad \ldots\ldots\ldots\ldots (1)$$

and the corresponding loss of head $= (v_1 - v_2)^2 \div 2g$, which, since $F_1 v_1 = F_2 v_2$ may be written

$$\left.\begin{array}{l}\textbf{LOSS OF HEAD IN}\\ \textbf{SUDDEN ENLARGEMENT}\end{array}\right\} = \left[\frac{F_2}{F_1} - 1\right]^2 \frac{v_2^2}{2g} \quad \ldots\ldots\ldots (2)$$

That is the co-efficient ζ for a sudden enlargement is $\zeta = \left(\frac{F_2}{F_1} - 1\right)^2 \quad \ldots\ldots\ldots (3)$

F_1 and F_2 are the respective sectional areas of the pipes

NOTE. Practically the flow cannot always be maintained with full sections. In any case if we assume the pipes to be running full (once started so) and on that assumption compute the internal pressure at F_1 and it zero or negative, the assumption is incorrect. That is unless there is some pressure at F_1 the water will not swell out laterally to fill the large pipe.

Example. Fig. 564. In the short tube AB containing a sudden enlargement, we have given $F_2 = F_m = 6$ sq. inches,

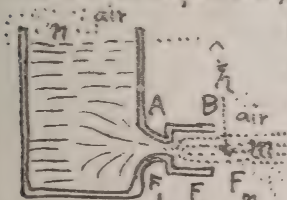


Fig. 564

$F_1 = 4$ sq. inches, and $h = 9$ feet. Required the velocity of the jet at m (in the air, so that $p_m \div \gamma = b = 34$ ft.) if the only loss of head considered is that due to the sudden enlargement (skin-friction neglected as the tube is short; the reservoir entrance has rounded corners) Applying B's theorem § 474 to m as down-stream section, and n in reservoir surface as up-stream position (m = datum-level) we have

$$\frac{v_m^2}{2g} + b + 0 = 0 + b + h - \zeta \frac{v_2^2}{2g} \quad \ldots\ldots\ldots (4)$$

But here $v_2 = v_m$ $\therefore$ $(1+\zeta)\dfrac{v_m^2}{2g} = h$ .... (5)

From eq. 3, we have .... $\zeta = \left(\frac{6}{4}-1\right)^2 = 0.25$ and

finally ft. lb. sec. $\Big\}\cdots v_m = \sqrt{\dfrac{1}{1.25}}\sqrt{2\times 32.2\times 9} = 0.895\sqrt{2\times 32.2\times 9}$

$= 21.55$ ft. per sec. (The factor 0.895 might be called a coefficient of velocity for this case) The volume of flow per second $\therefore$ is $Q = F_m v_m = \frac{6}{144}\times 21.55 = 0.898$ cub. ft. per sec.

We have so far assumed that the water fills both parts of the tube, i.e., that the pressure p_1, at F_1, is $>$ zero, (see foregoing note). To verify this assumption, we compute p_1 by applying B's theorem to F_1 as a down-stream position and datum plane and n as an up-stream position and obtain (no loss of head between) $\dfrac{v_1^2}{2g} + \dfrac{p_1}{\gamma} + 0 = 0 + b + h - 0$ .... (6)

But since $F_1 v_1 = F_2 v_2$ we have $v_1^2 = \left(\frac{6}{4}\right)^2 v_2^2 = \left(\frac{6}{4}\right)^2 v_m^2$ and $\therefore$ the pressure ~~head~~ at F_1 (substituting from eqs above) is

$\dfrac{p_1}{\gamma} = b + h - \left(\frac{6}{4}\right)^2 \dfrac{h}{1+\zeta} = 34 + 9 - \frac{9}{4}\cdot\dfrac{9}{1+.25} = 27$ feet

and $\therefore$ $p_1 = \frac{27}{34}$ of $14.7 = 11.6$ lbs per sq. inch, which is > 0, $\therefore$ efflux with the tube full in both parts can be maintained under 9 ft. head.

If, with F_1 and F_2 as before (and $\therefore$ ζ) we put $p_1 = 0$, and solve for h, we obtain $h = 42.5$ ft. as the max. head under which efflux, with the large portion full, can be secured.

480. SHORT PIPE. SQUARE-EDGED INTERNALLY. This case, already treated in §§ 466 and 473, (see Fig. 565, a repetition of 549) presents a loss of head due to the sudden enlargement from the contracted section at m (whose sectional area may be put $= CF$, C being an unknown co-efficient, or ratio of contraction) to the full section F of the pipe. From § 473 we know that the loss of head due to the short pipe is $h_r = \zeta_E \dfrac{v_m^2}{2g}$, in which $(\alpha = 90°)$ $\zeta_E = 0.505$ while from Borda's formula, § 479,

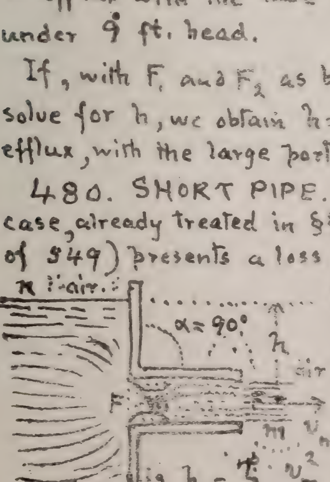


Fig. 565.

we have also $\zeta_E = \left(\frac{F}{CF} - 1\right)^2$. Equating these, we find the coefficient of internal contraction at m' to be

$$C = \frac{1}{1+\sqrt{\zeta_E}} = \frac{1}{1+\sqrt{.505}} = \frac{1}{1.711} = 0.584$$

or about 0.6 (compare with C = 0.64 for thin plate contraction §457). It is probably somewhat larger than this, since a small part of the loss of head, h_r, is due to friction at the corners and against the sides of the pipe.

By a method similar to that pursued in the example of §479 we may show that unless h is less than about 40 feet the tube can not be kept full, the discharge being as in Fig. 551. If the efflux takes place into a partial vacuum this limiting value of h is still smaller. Weisbach's experiments confirm these statements.

481. DIAPHRAGM IN A CYLINDRICAL PIPE. Fig. 566 The diaphragm being in "thin plate", let the circular opening in it have an area = F (concentric with pipe) while the section of the pipe is = F_2. Contraction occurs, to a contracted section F_1 = CF, in enlarging from which to the section F_2 of pipe, the stream suffers a loss of head

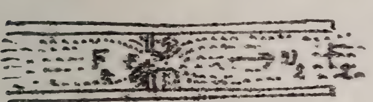


Fig. 566

which by Borda's formula is.....$h_r = \zeta \frac{v_2^2}{2g} = \left(\frac{F_2}{F_1} - 1\right)^2 \frac{v_2^2}{2g}$ where v_2 = velocity in pipe (supposed running full). Of course F_1 = CF depends on F but since experiments are necessary at any event, it is just as well to give the values of ζ itself, as determined by Weisbach's experiments, viz.:

For $F:F_2$ =	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
ζ =	226.	48.	17.5	7.8	3.7	1.8	.8	.3	.06	0.00

By internal lateral filling, Fig. 567, the change of section may be made gradual and eddying prevented, and then no loss of head (i.e. no loss of energy) is incurred, except the slight amount due to skin friction along this small surface.

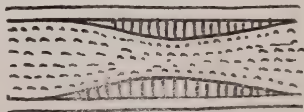
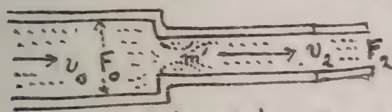


Fig. 567

482. SUDDEN DIMINUTION OF CROSS-SECTION. SQUARE EDGES. Fig. 568. Here, again, the resistance is due to the sudden enlargement from the contraction at m' to the full section F_2 of the small pipe, so that in



the loss of head, by Borda's formula, $h_r = \zeta \dfrac{v_2^2}{2g} = \left[\dfrac{F_2}{F_1} - 1\right]^2 \dfrac{v_2^2}{2g}$ ..(1)

The co-efficient

$$\zeta = \left(\frac{F_2}{F_1} - 1\right)^2 = \left(\frac{F_2}{CF_2} - 1\right)^2 = \left(\frac{1}{C} - 1\right)^2 \ldots\ldots (2)$$

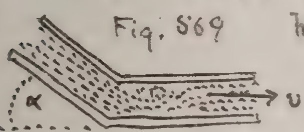
depends on the co-efficient of contraction C, but this latter is influenced by the ratio of F_2 to F_0 the sectional area of the larger pipe, C being about .60 when F_0 is very large, i.e. when the small pipe issues directly from a large reservoir so that $F_2 : F_0$ practically $= 0$. For other values of this ratio Weisbach gives the following table for C, from his own experiments.

For $F_2 : F_0 =$	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
$C =$	.624	.632	.643	.659	.681	.712	.755	.813	.892	1.00

C being found we compute ζ from eq. (2) for use in eq. (1)

483. ELBOWS. The internal disturbance caused by an elbow, Fig. 569 (pipe full both sides of elbow) occasions a loss

Fig. 569



$h_r = \zeta \dfrac{v^2}{2g}$ ......(1) in which, according to Weisbach's experiments with tubes 3 centims., i.e. 1.2 in. in diameter, we may put

for $\alpha =$	20°	40°	60°	80°	90°	100°	110°	120°	130°	140°
$\zeta =$	.046	.139	.364	.740	.984	1.26	1.556	1.86	2.16	2.43

computed from the empirical formula $\zeta = .9457 \sin^2 \frac{1}{2}\alpha + 2.047 \sin^4 \frac{1}{2}\alpha$

v is the velocity in pipe, α as in figure. For larger pipes ζ would probably be somewhat smaller.

If the elbow is immediately succeeded by another in the same plane and turning the same way, Fig. 570, the loss of head is not materially increased, since the eddying takes place

chiefly in the further branch of the second elbow; but if it

Fig. 570

turns in the reverse direction, Fig. 571, but still in the same plane, the total loss of head is double that of one elbow; while if the plane of the second is ⊥ to that of the first, the total loss of head is $1\frac{1}{2}$ times that of one alone. (Weisbach)

Fig. 571

484. BENDS IN PIPES OF CIRCULAR SECTION.

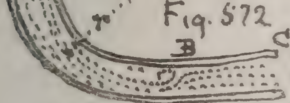


Fig. 572

Fig. 572. Weisbach bases the following empirical formula for ζ, the coefficient of resistance of a quadrant bend in a pipe of circular section, on his own experiments and some of Du Buat, viz.

$$\zeta = 0.131 + 1.847\left(\frac{a}{r}\right)^{7/2} \ldots\ldots(1), \text{ for use in } h_r = \zeta\frac{v^2}{2g} \ldots\ldots(2)$$

where a = radius of pipe, r = rad. of bend (to centre of pipe), and v = velocity in pipe. h_r = loss of head due to bend

It is understood that the portion BC of the pipe is kept full by the flow, which however may not be practicable unless BC is more than three or four times as long as wide and is full at the start. A semi-circular bend occasions about the same loss of head as a quadrant bend, but two quadrants forming a reverse curve in the same plane, Fig. 573, occasion a double loss. By enlarging the pipe at the bend, or providing internal thin partitions ‖ to the sides, the loss of head may be considerably diminished. Weisbach gives the following table, computed from eq. (1):

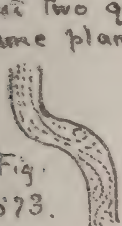


Fig. 573.

For $\frac{a}{r}$ =	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.0
ζ =	.131	.138	.158	.206	.294	.440	.661	.977	1.40	1.98

484. VALVE-GATES IN CYLINDRICAL PIPES.
THROTTLE VALVES " " " "

Adopting, as usual, the form $h_r = \zeta\frac{v^2}{2g} \ldots\ldots(1)$

~~for the~~ loss of head due to a valve-gate, Fig. 574

Fig. 574 Valve Gate.

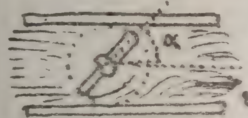


Fig. 575 THROTTLE VALVE

or for a throttle valve, each in a definite position, Weisbach's experiments furnish us with a range of values of ζ, in the case of a valve-gate or throttle valve, in a cylindrical pipe 1.6 in. in diameter, as follows: (for s, d, and α see figures.) v is the velocity in the full section of pipe, running full both sides.

Valve-gate		Throttle valve	
$s:d$	ζ	α	ζ
1.0	.00	5°	.24
7/8	.07	10°	.52
6/8	.26	15°	.90
5/8	.81	20°	1.54
4/8	2.06	25°	2.51
3/8	5.52	30°	3.91
2/8	17.0	35°	6.22
1/8	97.8	40°	10.8
		45°	18.7
		50°	32.6
		55°	58.8
		60°	118.0
		65°	256.
		70°	751.

455. EXAMPLES INVOLVING DIVERS LOSSES OF HEAD. We here suppose as usual, that the pipes are full during the flow. Practically, provision must be made for the escape of the air which collects at the high points. If this air is at a tension > one atmos., automatic air-valves will answer to allow its escape; if < one atmos. an air-pump must be used.

Example 1. Fig. 576. What head h will be required to deliver $\frac{1}{2}$ U.S. gallon

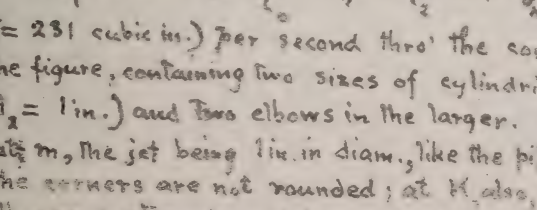


Fig. 576

(= 231 cubic in.) per second thro' the continuous line of pipe in the figure, containing two sizes of cylindrical pipe ($d_0 = 3$ in., $d_2 = 1$ in.) and two elbows in the larger. The flow is into the air at m, the jet being 1 in. in diam., like the pipe. At E, $\alpha = 90°$ and the corners are not rounded; at K also, square. Take the ft. lb. sec. system, in which $g = 32.2$

Since $Q = \frac{1}{2}$ gal. $= \frac{1}{2} \cdot \frac{231}{1728} = .0668$ cub. ft. per sec., ∴ the veloc. of jet is $v_m = v_2 = Q \div \frac{1}{4}\pi\left(\frac{1}{12}\right)^2 = 12.25$ ft. per sec.

∴ veloc. in large pipe is to be $v_0 = \left(\frac{1}{3}\right)^2 v_2 = 1.36$ " "

From Bernoulli's Theorem, § 474, we have (m = datum-level)

$$\frac{v_m^2}{2g} + b + 0 = \underbrace{0 + b + h - \zeta_E \frac{v_0^2}{2g} - 4f_0 \frac{l_0}{d_0}\frac{v_0^2}{2g} - 2\zeta_{el}\frac{v_0^2}{2g} - \zeta_K \frac{v_2^2}{2g} - 4f_2 \frac{l_2}{d_2}\frac{v_2^2}{2g}}_{eq.\ (1)}$$

involving six separate losses of head, for each of which there is no difficulty in finding the proper ζ or f, since the velocities and dimensions are all known, by consulting preceding paragraphs.

From § 473, table, for $\alpha = 90°$, we have .... $\zeta_E = 0.505$
From § 474a, for $d_0 = 3^{in}$, and $v_0 = 1.36$ ft. per sec. $f_0 = .00725$
" " " " $d_2 = 1^{in}$, " $v_2 = 12.25$ " " " $f_2 = .00613$
" § 483, elbows, for $\alpha = 90°$ .... ζ_{el} ... $= .984$
" § 482 for sudden diminution at K we have,
[since $F_2 \div F_0 = 1^2 \div 3^2 = 0.111$, ∴ $C = 0.625$] $\zeta_K = \left(\frac{1}{.625} - 1\right)^2 = 0.360$

Solving eq. (1), then, for h, and substituting (ft. lb. sec.) numerical values as above (noting that $v_m = v_2$ and $v_0 = \frac{1}{9}v_2$),

$$h = \frac{(12.25)^2}{2 \times 32.2}\left[1 + \left(\frac{1}{9}\right)^2\left(.505 + 4 \times .00725 \times \frac{50}{1/4} + 2 \times .984\right) + .360 + 4 \times .00613 \times \frac{20}{1/12}\right]$$

$$= \frac{(12.25)^2}{2 \times 32.2}\left[1 + \underbrace{(.00623 + .07160 + .0243)}_{\text{refer to losses in large pipe}} + \underbrace{(.360 + 5.8848)}_{\text{in small pipe}}\right]$$

∴ $h = 2.323 \times 7.3469 = 17.09$ feet; Ans.

It is here noticeable how small are the losses of head in the large pipe, the principal reason of which being that the velocity in it is so small ($v_0 = 1.36$ ft. per sec.) and that in general losses of head depend on the square of the velocity (nearly). In other words, the large pipe approximates to being a reservoir in itself.

With no resistances $h = v_m^2 \div 2g = 2.32^{ft.}$ would be sufficient head.

Example 2. Fig. 577. With the valve-gate V half drawn up (i.e. $s = \frac{1}{2}d$ in Fig. 574) required the volume delivered per second thro' the pipe there shown. The jet issues from a short straight pipe (of diam. $d_2 = 1\frac{1}{2}$ in.)

set with square corners in the end of the large one. Dimensions as in figure. Radius of each bend $= r = 2$ inches. The

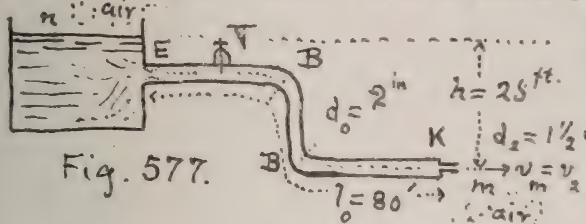


Fig. 577.

velocity v_m of jet in the air $= v_2$ that in small pipe; hence the loss of head at K $= \zeta_K \frac{v_2^2}{2g} = \zeta_K \frac{v_m^2}{2g}$

Now v_m is unknown, as yet, but v_0, the velocity in large pipe $= v_m \times \left[(\frac{3}{2})^2 : 2^2\right] = \frac{9}{16} v_m$. From B's theorem § 474, with m as datum level, we obtain, after transposition,

$$h = \frac{v_m^2}{2g} + \zeta_E \frac{v_0^2}{2g} + \zeta_V \frac{v_0^2}{2g} + 2\zeta_B \frac{v_0^2}{2g} + 4 f_0 \frac{l_0}{d_0} \frac{v_0^2}{2g} + \zeta_K \frac{v_m^2}{2g} \quad \ldots\ldots (1)$$

Of the co-efficients concerned, f_0 alone depends on the unknown velocity v_0. For the present (1st approx.) put $f_0 = .006$

From § 473, with $\alpha = 90°$ .... $\zeta_E = .505$
From § 484a, valve-gate with $s = \frac{4}{8} d$, ... $\zeta_V = 2.06$
From § 484, with $a : r = 0.50$ .... $\zeta_B = 0.294$
while at K, from § 482, having
$F_2 : F_0 = (\frac{3}{2})^2 : 2^2 = \frac{9}{16} = 0.562$, whence in table, $C = 0.700$
$\therefore \zeta_K = (\frac{1}{.70} - 1)^2 = (0.428)^2 =$ ..... $\zeta_K = 0.183$

Substituting in eq. (1) above, with $v_0^2 = (\frac{9}{16})^2 v_m^2$, we have

$$v_m = \sqrt{\frac{1}{1 + \frac{81}{256}\left[\zeta_E + \zeta_V + 2\zeta_B + \frac{4 f_0 l_0}{d_0}\right] + \zeta_K}} \sqrt{2gh} \quad \ldots (2)$$

in which the first radical, an abstract number, might be called a coefficient of velocity, ϕ, for the whole delivery pipe, and also, since $Q = F_m v_m = F_2 v_m = \mu F_2 \sqrt{2gh}$, it may be named a co-efficient of efflux, μ. With above numbers

$$v_m = \sqrt{\frac{1}{1 + \frac{81}{256}\left[.505 + 2.06 + 2 \times .294 + \frac{4 \times .006 \times 80}{1/6}\right] + .183}} \sqrt{2 \times 32.2 \times 25}$$

$\therefore v_m = 0.421\sqrt{2gh} = 0.421\sqrt{2 \times 32.2 \times 25} = 16.89$ ft. per sec.
(The .421 might be called a coef. of veloc.) $\therefore$ the vol. delivered per sec. is $Q = \frac{1}{4}\pi d_2^2 v_m = \frac{1}{4}\pi\left(\frac{3}{24}\right)^2 16.89 = .207$ cub. ft. per sec.
(As the section of the jet $F_m = F_2$ that of the short pipe or nozzle we might say that $.421 = \mu =$ co-ef. of efflux, for we may write $Q = \mu F_2\sqrt{2gh}$, $\therefore \mu = .421$)

486. TIME OF EMPTYING VERTICAL PRISMATIC VESSELS (OR <u>INCLINED</u> PRISMS IF BOTTOM IS HORIZONTAL) UNDER VARIABLE HEAD. *Case I.* Through an orifice or short pipe in the bottom. Fig. 578. As the upper free surface, of area $= F'$, sinks, F' remains constant. Let z = head of water at any stage of the emptying; it $= z_0$ at the outset, and $= 0$ when the vessel is empty. At any instant, Q, the rate of discharge (= vol. per time-unit) depends on z and is

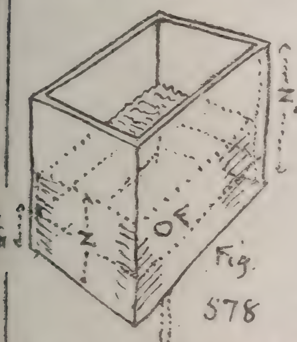


Fig. 578

$$Q = \mu F\sqrt{2gz} \quad \ldots\ldots\ldots (1)$$

where μ = co-ef. of efflux $= \phi C' =$ co-ef. of veloc. $\times$ co-ef. of contraction (see §454 eq. (3)). We here suppose F' so large compared with F the area of the orifice, that the free surface of the water in the vessel does not acquire any notable velocity at any stage, and that hence the rate of efflux is the same at any instant as for a steady flow with head $= z$, and a zero veloc. in free surface. μ is considered constant. From (1) we have

$$dV = (\text{vol. discharged in time } dt) = Q\,dt = \mu F\sqrt{2gz}\,dt \ldots (2)$$

But this is also equal to the vol. of the horiz. lamina, $F'dz$, thro' which the free surface has sunk in the same time dt.

$$\therefore -F'dz = \mu F\sqrt{2g}\,z^{\frac{1}{2}}dt, \quad \therefore \quad dt = \frac{-F'}{\mu F\sqrt{2g}} z^{-\frac{1}{2}}dz \ldots (3)$$

We have written <u>minus</u> $F'dz$ because, dt being an increment, dz is a decrement. To reduce the depth from z_0 (at the start, time $= t =$ zero) to z_n demands a time

$$\left[{}_0^n t = -\frac{F'}{\mu F\sqrt{2g}}\int_{z_0}^{z_n} z^{-\frac{1}{2}}dz = \frac{2F'}{\mu F\sqrt{2g}}\left[z_0^{\frac{1}{2}} - z_n^{\frac{1}{2}}\right] \ldots (4)$$

whence, by putting $z_1 = 0$, we have the time necessary

to empty the whole prism ..... eq. (5') $\left[{}_{0}^{\text{vol}}\right.$ $t = \dfrac{2F' z_0^{\frac{1}{2}}}{\mu F\sqrt{2g}} = \dfrac{2F' z_0}{\mu F\sqrt{2gz_0}} = \dfrac{2\,\text{Vol. of vessel}}{\text{initial rate of disch.}}$

that is, to empty the vessel requires double the time of discharging the same amount of water if the vessel had been kept full; (at constant head $= z_0 =$ altitude of prism)

To *fill* the same vessel through an orifice in the bottom, the flow through which is supplied from a body of water of infinite extent horizontally, as with the (single) canal lock of Fig. 579, will obviously require the same time as given in eq. (5) above, since the effective head z diminishes from z_0 to 0 according to the same law.

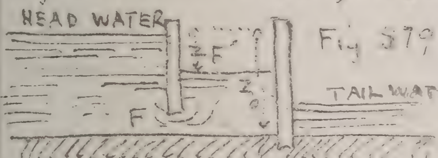


Fig. 579

Example. What time will be needed to empty a parallelopipedical tank (Fig. 578) 4 ft by 5 ft. in horiz. section and 6 ft deep, thro' a stop-cock in the bottom whose coefficient of efflux when fully open is known to be $\mu = 0.640$, and whose section of discharge is a circle of diam. $= \frac{1}{2}$ in.? From given dimensions $F' = 4 \times 5 = 20$ sq. ft., while $z_0 = 6$ ft $\therefore$ from eq. (5') (ft. lb. sec.)

$$\left.\begin{array}{l}\text{Time of}\\ \text{emptying}\end{array}\right\} = \frac{2 \times 20 \times \sqrt{6}}{0.64 \times \frac{1}{4}\pi\left(\frac{1}{24}\right)^2\sqrt{2 \times 32.2}} = \left\{\begin{array}{l}13620 \text{ seconds}\\ = 3^{\text{hours}}\ 47^{\text{min.}}\ 0^{\text{sec.}}\end{array}\right.$$

Case II. Two communicating prismatic vessels. Required the time to come to a common level ON, Fig. 580, efflux taking place thro' a small orifice, of area $= F$, under water. At any instant the rate of discharge is $Q = \mu\sqrt{2gz}$, as before. $z =$ difference of level. Now if F' and F'' are the horizontal sections of the two prismatic vessels (axes vertical) we have $F'x = F''y$

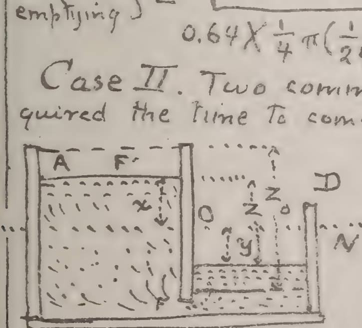


Fig. 580.

and $\therefore$ z, which $= x + y$, $= x + (F' \div F'')x$

$\therefore$ $x = z \div \left[1 + \frac{F'}{F''}\right]$ and $dx = dz \div \left[1 + \frac{F'}{F''}\right]$

As before, we have

$$-F' dx = \mu F \sqrt{2g}\; z^{1/2} dt \;\therefore\; dt = -\frac{F'F''}{F'+F''}\,\frac{z^{-1/2}dz}{\mu F\sqrt{2g}}$$

Hence, integrating, the Time for the diff. of level to change from z_0 to z_n $\Big\} = \frac{2F'F''}{F'+F''}\cdot\frac{z_0^{1/2} - z_n^{1/2}}{\mu F\sqrt{2g}}$ ......(6)

and, by making $z_n = 0$, Time of coming to a common level $\Big\} = \frac{2F'F''}{F'+F''}\cdot\frac{1}{\mu F}\sqrt{\frac{z_0}{2g}}$ .....(7)

Algebraic Example. In the double lock in Fig. 581,

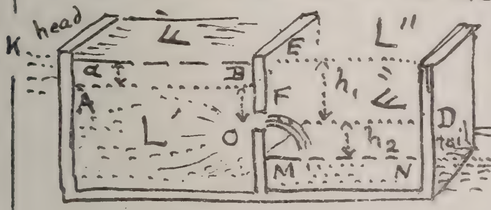


Fig. 581.

let L′ be full, while in L″ the water stands at a level MN = that of tail water. F′ and F″ are the horizontal sections of the prismatic locks. Let the orifice, F, between them, be at a depth $= h_1$ below the initial level KE of L′, and a height $= h_2$ above that, MN, of L″. The orifice at O, area = F, being opened, efflux from L′ begins into the air and the level of L″ is gradually raised from MN to OD, while that of L′ sinks from KE to AB a distance = a, computed from the relation Vol. $F'a$ = vol. $F''h_2$, and the time occupied is....

(see eq. 4) ..... $t_1 = \frac{2F'}{\mu F\sqrt{2g}}\left[\sqrt{h_1} - \sqrt{a}\right]$ ....(8)

As soon as O is submerged, efflux takes place under water and we have an instance of Case II. Hence the time of reaching a common level (after submersion of O) (see eq. 7) is

$$t_2 = \frac{2F'F''}{\mu F(F'+F'')}\sqrt{\frac{h_1 - a}{2g}} \;\;......(9)$$

and the total Time is $= t_1 + t_2$, with $a = F''h_2 \div F'$.

Case III. Emptying a vertical prismatic vessel thro' a rectangular "notch" in the side, or overfall. Fig. 582. As before, let even the initial area ($= z_0 b$) of the notch be small compared with the horizontal area F' of tank. Let z = depth of lower sill of notch below level of tank surface at any instant, and b = width of notch. Then, at any in-

stant (see eq. 10 §464)

Rate of disch. (vol.) $= Q = \frac{2}{3}\mu_0 bz\sqrt{2gz} = \frac{2}{3}\mu b\sqrt{2g}\, z^{3/2}$

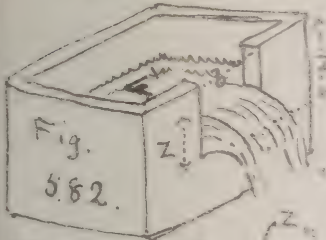


Fig. 582.

$\therefore$ disch. in $dt = \frac{2}{3}\mu b\sqrt{2g}\, z^{3/2}dt$ and putting this $= -F'dz =$ vol. of water lost by the tank in time dt

$\therefore dt = -\frac{3}{2}\frac{F'}{\mu b\sqrt{2g}}z^{-\frac{3}{2}}dz$, whence

$$\left[_0^n t = -\frac{3}{2}\frac{F'}{\mu b\sqrt{2g}}\int_{z_0}^{z_n} z^{-\frac{3}{2}}dz = -\frac{3}{2}\frac{F'}{\mu b\sqrt{2g}}\left[_0^n -\frac{1}{\frac{1}{2}}z^{-\frac{1}{2}} = \frac{3F'}{\mu b\sqrt{2g}}\left[\frac{1}{\sqrt{z_n}}-\frac{1}{\sqrt{z_0}}\right]\right.\right.$$

as the time in which the tank surface sinks from a height z_0 above sill to a height z_n above sill. If we inquire the time t'' for the water to sink to the level of the sill of the notch we put $z_n =$ zero, $\therefore t'' =$ infinity. In explanation of this result note that as z diminishes, not only does the velocity of flow diminish but the available area of efflux ($= zb$) also grows less, whereas in Cases I and II the orifice of efflux remained of constant area $= F$

486a. TIME OF EMPTYING VESSELS OF VARIABLE HORIZONTAL SECTIONS. Considering regular geometrical forms first, let us take *Case I. Wedge-shaped vessel*, edge horizontal and underneath, orifice in the edge, so that z the variable head is always the altitude of a triangle similar to the section ABC of the body of water when efflux begins. At any instant during the efflux the area, S, of the free surface, being variable here, takes the place of F' in eq. 3 of §485 whence,

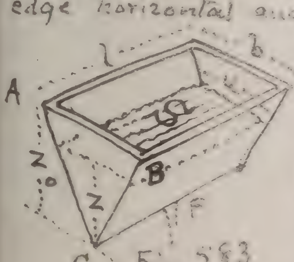


Fig. 583

(FOR ANY CASE OF VARIABLE FREE SURFACE) $dt = \dfrac{-Sz^{-\frac{1}{2}}dz}{\mu F\sqrt{2g}}$ ..(10)

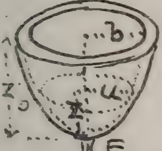
In the present case $S = ul$, and from similar triangles, $u : z :: b : z_0$ $\therefore S = blz \div z_0$, and $\therefore$

$$dt = \frac{-blz^{1/2}dz}{\mu F z_0\sqrt{2g}} \therefore \left[_0^n t = \frac{bl}{\mu F z_0\sqrt{2g}}\int_{z_0}^{z_n} z^{1/2}dz = \frac{\frac{2}{3}bl}{\mu F z_0\sqrt{2g}}\left[z_0^{3/2} - z_n^{3/2}\right]\right.$$

∴ To empty whole vessel, (put $z_n = 0$) } time $= \frac{4}{3} \cdot \frac{\frac{1}{2} b l z_0}{\mu F \sqrt{2 g z}} = \frac{4}{3} \cdot \frac{\text{Vol. of wedge}}{\text{initial rate of disch.}}$

i.e. $\frac{4}{3}$ as long as to discharge the same volume of water under a constant head $= z_0$ { Also true when ends are oblique.

Case II. Right segment of paraboloid of revolution Fig. 584. Axis vertical. Orifice at vertex. Here the variable



free surface has at any instant an area $= S = \pi u^2$, u being the radius of the circle and variable. From a property of the parabola $u^2 : b^2 :: z : z_0$ ∴ $S = \pi b^2 z \div z_0$

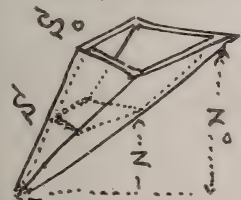
Fig. 584 ∴ (see eq. 10) $dt = \frac{-\pi b^2 z^{1/2} dz}{\mu F z_0 \sqrt{2g}}$ and hence

$$\Big[t\Big]_0^n = \frac{\pi b^2}{\mu F z_0 \sqrt{2g}} \int_{z_n}^{z_0} z^{1/2} dz = \frac{2}{3} \frac{\pi b^2}{\mu F z_0 \sqrt{2g}} \left[z_0^{3/2} - z_n^{3/2} \right];$$ while to

empty the vessel, the time (${z_n = 0}$) $= \frac{4}{3} \cdot \frac{\pi b^2 \frac{1}{2} z_0}{\mu F \sqrt{2 g z_0}} = \frac{4}{3} \cdot \frac{\text{total vol.}}{\text{initial rate of disch}}$

Same result as for the wedge, in Case I; in fact applies to any vessel in which areas of horizontal sections vary directly with their heights above orifice.

Case III. Any pyramid or Cone; vertex down; small orifice in vertex Fig. 585. Let area of



the base $= S_0$, at upper edge of vessel. At any stage of the flow S = area of base of pyramid of water. From similar pyramids $S_0 : S :: z_0^2 : z^2$ ∴ $S = \frac{S_0}{z_0^2} z^2$

Fig. 585 ∴ from eq. 10, $dt = -\frac{S_0}{z_0^2} \frac{1}{\mu F \sqrt{2g}} z^{3/2} dz$

to empty, time ($z_n = 0$) } $= \frac{S_0}{\mu F z_0^2 \sqrt{2g}} \int_0^{z_0} z^{3/2} dz = \frac{2}{5} \frac{S_0 z_0^{5/2}}{\mu F z_0^2 \sqrt{2g}} = \frac{6}{5} \cdot \frac{\text{Tot. Vol}}{\text{init. rate}}$

Case IV. Sphere. Similarly we may show that to empty a sphere, of rad. $= r$, thro' a small orifice F in lowest part } time $= \frac{16 \pi r^3}{15 \mu F \sqrt{g r}}$

487. TIME OF EMPTYING AN OBELISK-SHAPED VESSEL. A reservoir having this form (an obelisk may be defined as a solid of six plane faces two of which are rectangles in ∥ planes and with sides respectively ∥, the others trapezoids; a frustum of a pyramid is a particular case) is of common occurrence; see Fig. 585. Let the altitude $= h$, and

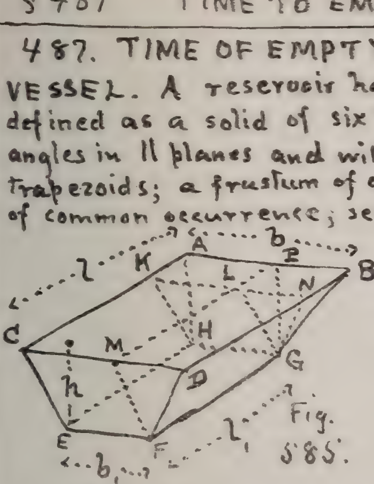


Fig. 585.

the two rectangular faces horizontal, with dimensions as in figure. By drawing thro' F, G, and H, right lines ∥ to EC, to cut the upper base, we form a rectangle KLMC equal to the lower base. Produce ML to P and KL to N and join PG and NG. We have now subdivided the solid into a parallelopiped KLMC−EHGF, a pyramid PBNL−G, and two wedges viz.: APLK−HG and LNDM−FG, with their edges horizontal; and may obtain the time necessary to empty the whole obelisk-volume by adding the times which would be necessary to empty the individual component volumes, separately, thro' the same orifice or pipe in the bottom plane EG. These have been already determined in §§ 485, 486, & 486a. The dimensions of each component volume may be expressed in terms of those of the obelisk, and all have a common altitude $= h$.

Assuming the orifice to be in the bottom, or else that the discharging end of the pipe, if such is used, is in *the plane of the bottom* EG, we have as follows, F being the area of discharge

Time to empty the parallelopiped separately would be (Case I § 485) $\Big\}\ldots t_1 = \dfrac{2b_1 l_1}{\mu F\sqrt{2g}}\sqrt{h}$ ...(1)

Time to empty the two wedges separately, Case I § 486a, $\Big\}\ldots t_2 = \dfrac{2}{3}\cdot\dfrac{b_1(l-l_1)+l_1(b-b_1)}{\mu F\sqrt{2g}}\sqrt{h}$.(2)

For the pyramid Case III § 486a $\Big\}\ldots t_3 = \dfrac{2}{5}\cdot\dfrac{(l-l_1)(b-b_1)}{\mu F\sqrt{2g}}\sqrt{h}$ ..(3) Hence to empty the whole

reservoir we add t_1, t_2, and t_3 $\Big\}\ldots t = \Big[3bl+8b_1l_1+2bl_1+2b_1l\Big]\dfrac{2\sqrt{h}}{15\mu F\sqrt{2g}}$ ....(4)

Example. Let a reservoir of above form, and with $b =$ 50 ft. $l =$ 60 ft, $b_1 =$ 10 ft, $l_1 =$ 20 ft., and depth of water $h = 16$ ft,

be emptied through a straight iron pipe, *horizontal*, and leaving the side of the reservoir close to the bottom, at an angle $\alpha = 36°$ with the inner plane of side. The pipe is 80 ft. long and 4 inches in internal diameter; and of clean surface. The jet issues directly from this pipe into the air and hence $F = \frac{1}{4}\pi(\frac{1}{3})^2$ sq. feet. To find μ the "coeff. of efflux" (= ϕ the co-eff. of velocity in this case since there is no contraction at disch. orif.) we use eq. 4 (the first radical), with f approx. = .006, and obtain $\mu = \phi = \sqrt{\dfrac{1}{1+\zeta_\varepsilon + 4f\frac{l}{d}}} = \sqrt{\dfrac{1}{1+.896+\frac{4\times.006\times80}{1/3}}}$
$= 0.361$

(N.B. Since the velocity in the pipe diminishes from a value $v = .361\sqrt{2g\times16} = 11.6$ ft. per sec. at the beginning of the flow to $v =$ zero at the close, $f = .006$ is a reasonable approximate average with which to compute the average ϕ above; see §474a)

Hence from eq. 4 of this § (ft. lb. sec. system)

$$t = \frac{[3\times50\times60 + 8\times10\times20 + 2(50\times20+20\times60)]\,2\sqrt{16}}{15\times0.361\times\frac{\pi}{4}(\frac{1}{3})^2\sqrt{2\times32.2}}$$

$\doteq 31630$ sec. $\doteq$ 8 hrs. 47 min. 10 sec. $\left\{\begin{array}{l}\text{Probably within 2}\\ \text{or 3\% of the Truth.}\end{array}\right.$

488. TIME OF EMPTYING RESERVOIRS OF IRREGULAR SHAPE. SIMPSON'S RULE. From eq. 10 §486 we have for the time in which the free surface of water in a vessel of any shape whatever sinks thro' a vertical distance $= dz$

$$dt = \frac{-Sz^{1/2}dz}{\mu F\sqrt{2g}} \quad \text{whence} \left[\text{time}\right]_{z=z_0}^{z=z_n} = \frac{1}{\mu F\sqrt{2g}}\int_{z_n}^{z_0} Sz^{-\frac{1}{2}}dz \quad \ldots(1)$$

where S is the variable area of the free surface at any instant and z the head of water at the same instant, efflux proceeding thro' a small orifice (or extremity of pipe) of area $= F$. If S can be expressed in terms of z we ~~may~~ can integrate eq. (1) (i.e. provided that $Sz^{-1/2}$ has a known anti-derivative) but if not, the vessel or reservoir being irregular in form, as in Fig. 586, which shows a pond whose bottom has been accurately surveyed so that we know the value of S for any stage of the emptying, we can still get an approximate solution by using Simpsons Rule for approximate integration

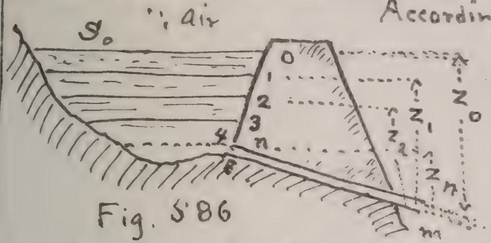


Fig. 586

Accordingly, if we inquire the time in which the surface will sink from O to the entrance E of the pipe in Fig. 586 (or to any point short of that) we divide the vertical distance from O to n (4 in this figure) into an even number of equal parts and from the known form of the pond compute the area S corresponding to each point of division, calling them S_0, S_1, etc. Then the required time is approximately

$$\left[_0^n t = \frac{z_0 - z_n}{\mu F \sqrt{2g}\, n}\left[\frac{S_0}{z_0^{1/2}} + 4\left(\frac{S_1}{z_1^{1/2}} + \frac{S_3}{z_3^{1/2}} + \cdots\right) + 2\left(\frac{S_2}{z_2^{1/2}} + \frac{S_4}{z_4^{1/2}} + \cdots\right) + \frac{S_n}{z_n}\right]\right.$$

Example. Fig. 586. Suppose we have a pipe Em of same design as in the example of §487 and an initial head of $z_0 = 16$ ft., so that the same value of $\mu = .361$ may be used. Let $z_n - z_0 = 8$ feet, and divide this interval (of 8 ft.) into four equal vertical spaces of 2 ft. each. If at the respective points of division we find from a previous survey that $S_0 = 400\,000$ sq. ft.; $S_1 = 320\,000$ sq. ft; $S_2 = 270\,000$ sq. ft.; and $S_3 = 210\,000$, $S_4 = 180\,000$ " " ; while $n = 4$, $\mu = .361$, and the area $F = \frac{1}{4}\pi\left(\frac{1}{3}\right)^2 = .0873$ sq. ft; we obtain (ft. lb. sec.)

$$\left[_0^4 t = \frac{16-8}{0.361 \times .0873\sqrt{2 \times 32.2} \times 3 \times 4}\left[\frac{400000}{\sqrt{16}} + \frac{4 \times 320000}{\sqrt{14}} + \frac{2 \times 270000}{\sqrt{12}} + \frac{4 \times 210000}{\sqrt{10}} + \frac{180000}{\sqrt{8}}\right] = 2\,444\,000 \text{ sec.}\right.$$

$$= 28^{\text{days}}\ 6^{\text{hrs.}}\ 53^{\text{m}}\ 20^{\text{sec.}}$$

The volume discharged, V, may also be found by Simpson's Rule, viz: Since each infinitely small horizont. lamina has a volume $dV = -S\,dz$ $\therefore$ $\left[_0^n V = \int_n^0 S\,dz\right.$, and with $n = 4$ we have (ft-lb. sec)

$$\left[_0^n \text{Vol.} = \frac{16-8}{3 \times 4}\left[400000 + 4\begin{Bmatrix}320000\\+210000\end{Bmatrix} + 2 \times 270000 + 180000\right] = 2{,}160{,}000 \text{ cub. ft.}\right.$$

489. VOLUME OF IRREGULAR RESERVOIR DETERMINED BY OBSERVING PROGRESS OF EMPTYING. Trans-

forming eq. (10) § 486 we have $S\,dz = -\mu F\sqrt{2g}\, z^{1/2} dt$. But $S\,dz$ is the infinitely small volume dV of water lost by the reservoir in the time dt, so that the volume of the reservoir between the initial and final (0 and n) positions of the horizontal free surface (at beginning and end of the time t_n) may be written

$$\left[_0^n V = \mu F\sqrt{2g} \int_0^{t_n} z^{1/2} dt \quad \ldots\ldots(1)\right.$$

This can be integrated approx. by Simpson's Rule, if the whole time of emptying be divided into an even number of equal parts, and the values z_0, z_1, z_2, etc. of the head of water noted at these equal intervals of time (not of vertical height). The corresponding surface planes will not be equi-distant, vertically, in general.

Fig. 587.

Whence for the particular case when $n = 4$ (see fig. 587)

$$\left.\begin{array}{l}\text{Vol. be-}\\\text{tween planes } 0\ \&\ 4\end{array}\right\} = \frac{\mu F\sqrt{2g}\,(t_n - 0)}{3 \times 4}\left[z_0^{1/2} + 4\left\{\begin{array}{l} z_1^{1/2} \\ + z_3^{1/2}\end{array}\right\} + 2 z_2^{1/2} + z_4^{1/2}\right] \quad ..(2)$$

Chap. V. Hydrodynamics continued; STEADY FLOW OF WATER IN OPEN CHANNELS.

490. NOMENCLATURE. Fig. 588. When water flows in an open channel, as in rivers canals, mill-races, water-courses ditches, etc., the bed and banks being rigid, the upper surface is free to conform in shape to the dynamic conditions of each case, which ∴ regulate to that extent the shape of the cross section.

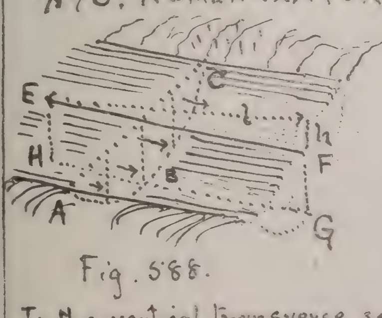


Fig. 588.

In the vertical transverse section AC in figure, the line AC is called the *air-profile* (usually to be considered horizontal and straight) while the line ABC, or profile of the bed and banks, is called the *wetted perimeter*. It is evident that

the ratio of the ~~whole~~ wetted perimeter to the whole perimeter, though never $< \frac{1}{2}$, varies with the form of the transverse section.

In a longitudinal section of the stream, EFGH, the angle made a surface filament EF with the horizontal is called the *slope*, and is measured by the ratio $s = h : l$, where l is the length of a portion of the filament and h = the *fall*, or vertical distance between the two ends of that length. The angle between the horizontal and the line HG along the bottom is not necessarily = that of the surface, unless the portion of the stream forms a prism; i.e., the slope of the bed does not necessarily = s = that of surface.

Examples. The old Croton Aqueduct has a slope of 1.10 ft. per mile, i.e. $s = .000208$. The new aqueduct (for New York) has a slope $s = .000132$, with a large transverse section. For large sluggish rivers s is much smaller.

491. VELOCITY MEASUREMENTS. Various instruments and methods may be employed for this object, some of which are:

Surface floats; are small balls, or pieces of wood, etc., so colored and weighted as to be readily seen, and still but little affected by the wind. These are allowed to float with the current in different parts of the width of the stream, and the surface velocity c in each experiment computed from $c = l \div t$, where l is the distance described between parallel transverse alignments (or actual ropes where possible) whose distance apart is measured on the bank, and t = the time occupied.

Double-floats. Two balls (or small kegs) of same bulk and condition of surface, one lighter, the other heavier than water, are united by a slender chain, their weights being so adjusted that

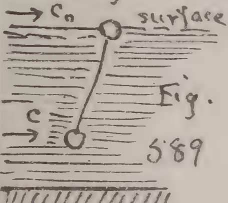


Fig. 589

the light ball, without projecting notably above the surface, buoys the other ball at any assigned depth. Fig. 589. It is assumed that the combination moves with a velocity c', equal to the arithmetic mean of the surface velocity c_0 of the stream and that, c, of the water filaments at the depth of the lower ball, which latter, c, is generally $< c_0$. That is, we have

$$c' = \tfrac{1}{2}(c_0 + c) \text{ and } \therefore \quad c = 2c' - c_0 \quad \ldots\ldots\ldots (1)$$

That is, c_0 having been previously obtained, eq. (1) gives the velocity c at any depth of the lower ball, c' being observed.

The floating staff is a hollow cylindrical rod, of adjustable length, weighted to float upright with the top just visible. Its observed velocity is assumed to be an average of the velocities of all the filaments lying between the ends of the rod.

Woltmann's Mill; or Tachometer; or Current-meter, Fig. 590, consists of a small wheel with inclined floats (or of a small "screw-propeller" wheel) held with its plane ⊥ to the current, which causes it revolve at a speed nearly proportional to the velocity c, of the water passing it. By a screw-gearing on the shaft, connection is made with a counting apparatus to record the number of revolutions. Sometimes a vane is attached, to compel the wheel to face the current. It is either held at the extremity of a pole or, by being adjustable along a staff fixed in the bed, may be set at any desired depth below the surface.

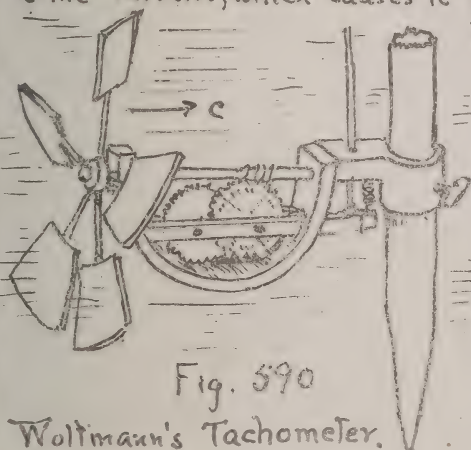


Fig. 590

Woltmann's Tachometer.

It is usually so designed as to be thrown in and out of gear by a cord and spring, see figure; that the time of making the indicated number of revolutions may be exactly noted.

By experiments in currents of known velocities a table or formula can be constructed by which to interpret the indications of any one instrument; i.e. to find the velocity c of the current corresponding to ~~a given~~ an observed number of revolutions per minute.

Pitot's Tube.... consists in principle of a vertical tube open above while its lower end, also open, is bent horizontally up stream; see A in figure. After the oscillations have ceased, the water in the tube remains stationary with its free

Fig. 591

surface a height, h', above that of the stream, on account of the continuous impact of the current against the lower end. By the addition of another vertical tube (see D in figure) with the face of its lower (open) end ∥ to the current (so that the water level in it is the same as that of the current), both tubes being provided with stopcocks, we may, after closing the stop-cocks, lift the apparatus into a boat and read off the height h' at leisure. We may also cause both columns of water to mount, thro' flexible tubes, into convenient tubes in the boat by putting the upper ends of both tubes in communication with a receiver of rarefied air, and thus watch the oscillations and obtain a more accurate value of h'. [See Van Nostrand's Mag. for Mar. '78 p. 255] Theoretically, the thickness of the walls of the tube at the lower end being considerable, we have

$$c = \sqrt{g h'} \quad \ldots (1)$$

as a relation between c, the velocity of the particles impinging on the lower end, and the static height h' (§ 514). Eq. (1) is verified fairly well in practice. For instance, Weisbach's experiments with a fine instrument, used with velocities of from 0.32 to 1.24 metres per second, found $c = [3.54\sqrt{h' \text{ in met.}}]$ metres per second, whereas eq. (1) gives $c = [3.133\sqrt{h' \text{ met.}}]$ met. per sec. Pitot's tube, though simple, is not so accurate as the tachometer.

The *Hydrometric Pendulum*, a rather uncertain instrument, is readily understood from Fig. 592. The side AB, of the quadrant ABC, being held vertical, the plane of the quadrant is made ∥ to the current. The angle θ between the cord and the vertical depends on the weight G of the ball (heavier than water) and the amount of P the impulse or horizontal pressure of the current against the latter, since the cord will take the direction of the resultant R, for equilibrium.

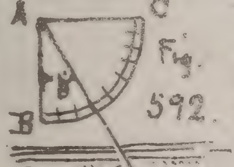


Fig. 592.

Now P (see § 519) for a ball of given size and character of surface, varies (nearly) as the square of the velocity; i.e. if P' is the impulse on a given stationary ball when the veloc. $= c'$ then for any other veloc. $P = \text{impulse} = \dfrac{P'}{c'^2}\, c^2 \quad \ldots\ldots (2)$

Hence, from this and the relation (equil.) ... $\tan\theta = P \div G$

$$c = \sqrt{Gc'^2 \div P'}\ \sqrt{\tan\theta} \quad \ldots\ldots\ldots\ldots (3)$$

With a given instrument and a <u>specified</u> <u>system</u> of units, the numerical value of the first radical may be determined as a single quantity, by experimenting with a known veloc. and the value of θ thus indicated, and may then, as a constant factor, be employed in (3) for finding the value of c for <u>any</u> observed value of θ; but the <u>same</u> <u>units</u> must be used as before.

492. VELOCITIES IN DIFFERENT PARTS OF A TRANSVERSE SECTION. The results of velocity-measurements made by many experimenters do not agree in supporting any very definite relation between the greatest surface velocity (c_0 max.) of a transverse section and the velocities at other points of the section, but establish a few general propositions.

1st In any vertical line the velocity is a maximum quite near the surface and diminishes from that point both toward the bottom and toward the surface.

2ndly In any horizontal line the velocity is a maximum near the middle of the stream, diminishing toward the banks.

3. The *mean velocity* $= v$, of the whole transverse section, i.e. the velocity which must be multiplied by the area, F, of the section, to obtain the vol. delivered per unit of time

$$\text{viz.}\quad Q = Fv \quad \ldots\ldots\ldots\ldots\ldots (1)$$

is about 83 per cent. of the max. surface velocity (c_0 max.) observed when the air is still, i.e.

$$v = 0.83 \times c_{0\,max.} \quad \ldots\ldots\ldots (2)$$

Of eight experimenters cited by Prof. Bowser, only one gives a value ($= 0.62$) differing more than .05 from .83, while others obtained the values .82, .78, .83, .80, .82, .83.

In the survey of the Mississippi River by Humphreys & Abbot, it was found that the law of variation of the velocity in any given vertical line could be fairly well represented by the ordinates of a parabola (Fig. 593) with its axis horizontal and its vertex at a distance d_1 below the surface according to the following relations, f'' being a number dependent on the force of the wind (from 0 for no wind to 10 for a hurricane)

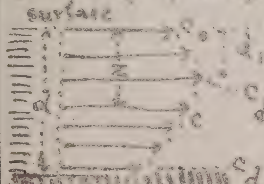


Fig. 593

$$d_1 = \left[0.317 \pm 0.06\, f''\right] d \quad \ldots\ldots (3)$$

where d is the total depth and the double sign is to be taken + for an up-stream wind, − for a down-stream. The following relations were also based on the results of the survey: (putting, for brevity, $B = 1.69 \div \sqrt{d+1.5}$) ....(4)

$$c = c_{d_1} - \sqrt{Bv}\left(\frac{z-d_1}{d}\right)^2 \;.....(5),\quad c_m = \frac{2}{3}c_{d_1} + \frac{1}{3}c_d + \frac{d_1}{d}\left(\frac{1}{3}c_0 - \frac{1}{3}c_d\right) \;..(6)$$

and $c_{\frac{1}{2}d} = c_m + \frac{1}{12}\sqrt{Bv}$ ....(7) ... UNITS FOOT AND SECOND

In these eq.s c = veloc. at any depth z below the surface
c_m = mean veloc. of the vertical curve.
c_{d_1} = max. " " " "
$c_{\frac{1}{2}d}$ = " at mid-depth { and c_d = vel. at bottom
and v = mean veloc. of whole section.

It was also found that the parameter of the parabola varied inversely as the square root of the mean veloc. c_m of curve.

In general the bottom velocity (c_d) is somewhat more than ½ the max. vel. (c_{d_1}) in the same vertical. In the Mississippi the veloc. at mid-depth in any vertical was found to be very nearly .96 of the surface vel. in same vertical; this fact is important, as it simplifies the approx. gauging of a stream.

493. GAUGING A STREAM OR RIVER. Where the relation (eq. 2 § 492) $v = .83\, c_{0\,max.}$ is not considered accurate enough for substitution in $Q = Fv$ to obtain the volume of discharge (or delivery) Q of a stream per time-unit, the transverse section may be divided into a number of subdivisions as in Fig. 594, of widths a_1, a_2, etc., and mean depths d_1, d_2, etc., and the respective mean velocities, c_1, c_2, etc. computed from measurements with current meters; whence we may write

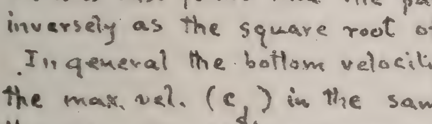
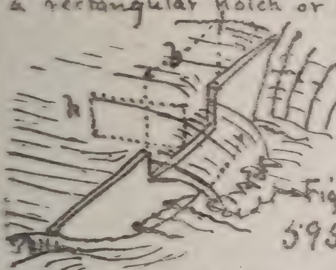


Fig. 594.

$$Q = a_1 d_1 c_1 + a_2 d_2 c_2 + a_3 d_3 c_3 + \text{etc.} \;.......(7)$$

With a small stream or ditch, however, we may erect a vertical boarding and allow the water to flow through a

a rectangular notch or overfall, Fig. 595, and after the head



surface has become permanent, measure h (depth of sill below the level surface somewhat back of boards) and b (width) and use the formulae of § 464; see Example in that article.

Fig. 595.

494. UNIFORM MOTION IN AN OPEN CHANNEL. We shall now consider a straight stream of indefinite length in which the flow is steady, i.e. a state of permanency exists, as distinguished from a freshet or a wave. That is, the flow is steady when the water assumes fixed values of mean veloc. v, and sectional area F, on passing a given point of the bed or bank; and the

EQ. OF CONTINUITY.... $Q = Fv = F_0 v_0 = F_1 v_1$ = constant ....(1)

holds good whether those sections are equal or not.

By UNIFORM MOTION is meant that, (the section of the bed and banks being of constant size and shape) the slope of the bed, the quantity of water (vol. = Q) flowing per time-unit, and the extent of the wetted perimeter, are so adjusted to each other that the mean velocity of flow is the same in all transverse sections and consequently the area and shape of the transverse section is the same at all points; and the slope of the surface = that of the bed. We may ∴ consider, for simplicity, that we have to deal with a prism of water of indefinite length sliding down an inclined rough bed of constant angle with uniform velocity, i.e. the mean veloc. = v common to all the sections (eq. 2 § 492) i.e. there is no acceleration. Let Fig. 596 show, free, a

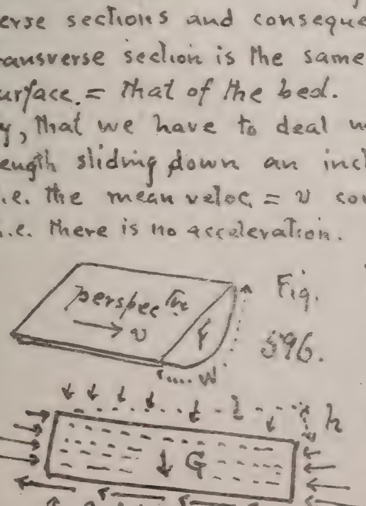


Fig. 596.

portion of this prism, of length = l and having its bases ⊥ to the bed and surface. The hydrostatic pressures at the two ends balance each other from the identity of conditions. The only forces having components ∥ to the bed and surface are the weight

$G = Fl\gamma$ of the prism (where γ = heaviness of water) making an angle = s (= slope) with a normal to the surface, and the friction between the water and the bed which is ∥ to the surface. The amount of this friction for the prism in question may be expressed as in § 469, viz.:

$$P = \text{fric.} = f\, S\gamma \frac{v^2}{2g} = f\, wl\gamma \frac{v^2}{2g} \ldots\ldots\ldots\ldots\ldots (2)$$

in which $S = wl$ = rubbing surface (area) = wetted perimeter, w, X length (see § 490), and f an abstract number (§ 469)

Since the mass of water in Fig. 596 is supposed to be in relative equilibrium, we may apply to it the laws of motion of a rigid body, and since the motion is a <u>uniform</u> <u>translation</u> (§ 109) the components, ∥ to the surface, of all the forces must balance.

$\therefore G \sin s$ must $= P$ = fric. $\therefore\ Fl\gamma \frac{h}{l} = f\, wl\gamma \frac{v^2}{2g}$;

whence $h = f\, \frac{wl}{F}\, \frac{v^2}{2g} \ldots (3)'$, or $h = f\, \frac{l}{R} \cdot \frac{v^2}{2g} \ldots (3)$

in which $F \div w$ is called R the *hydraulic mean depth*, or *hydraulic radius*. (3) is sometimes expressed by saying that the whole fall, or head, h, is in uniform motion absorbed in friction-head. Also since the slope $s = h \div l$ we also have

$$v = \sqrt{\frac{2g}{f}}\, \sqrt{Rs}\ ;\quad \text{or (not homog.)} \ldots \begin{cases} v = A\sqrt{Rs} \ldots (4) \\ \text{CHÉZY'S Formula} \end{cases}$$

The coefficient $A = \sqrt{2g \div f}$ is not, like f, an abstract number but its numerical value depends on the system of units employed.

The "coefficient of liquid friction", f, on river beds is somewhat greater than that in pipes (§ 474a) and according to Kutter and Ganguillet depends not only on the velocity v, but also on the slope s, on the value of R the mean hydraulic depth, and upon the nature (roughness) of the bed.

Eytelwein, as an average of 91 experiments, made $f = .007565$ for streams of moderate size. Weisbach makes f depend only on v; thus:

v ft. per sec.	f	v ft. per sec.	f	v ft. per sec.	f
0.4	.01097	1.0	.00883	5.	.00769
0.6	.00978	2.0	.00812	10.	.00755
0.8	.00918	3.0	.00788	15.	.00750

Kutter and Ganguillet, however, have recently (1869) proposed a formula which harmonizes in a single equation a great number of experimental data including those obtained in the survey of the Mississippi River. They make the co-efficient A in eq. (4) (i.e. $\sqrt{2g \div f}$) a function of R, s, and n an abstract number, or co-efficient of roughness, depending on the nature of the surface of the bed and banks, viz.

$$\underset{\left[\substack{\text{Feet and}\\ \text{sec., units}}\right]}{v^{\text{ft. per sec.}}} = \left[\frac{41.6 + \dfrac{1.811}{n} + \dfrac{.00281}{s}}{1 + \left(41.6 + \dfrac{.00281}{s}\right)\dfrac{n}{\sqrt{R^{\text{ft.}}}}}\right]\sqrt{R^{\text{ft.}}\, s} \quad \ldots\ldots(5)$$

KUTTER'S Formula.

Values of n. $n = .009$ for well planed timber bed;
.010 Plaster in pure cement, 0.011 for plaster in cement with $\frac{1}{3}$ sand.
.012 for unplaned timber .013 Ashlar and brickwork
.015 " canvas lining on frames .017 for rubble.
.020 Canals in very firm gravel.
.025 for rivers and canals in perfect order and regimen and perfectly free from stones and weeds.
.030 " " " " " {moderately good order and regimen having stones and weeds occasionally
.035 " " " " in bad order and regimen, overgrown with vegetation and strewn with stones or detritus of any sort.

Kutter's formula is claimed to apply to all kinds and sizes of watercourses from large rivers to sewers and ditches; for uniform motion. If $\sqrt{R}$ is the unknown quantity Kutter's formula leads to a quadratic equation; if s the slope, to a cubic. Hence, to save computation, tables have been prepared, some of which will be found in Vol. 28 (pp. 135 and 393) (sewers), and in Jackson's works on Hydraulics (rivers).

Example 1. A canal 1000 ft. long of the trapezoidal section in Fig. 597 is required to deliver 300 cubic ft of water per second with the water 8 ft deep at all sections (i.e. with uniform motion), the slope of the bank being such that for a depth of 8 ft. the width of the water surface (or length of air-profile) will be 20 ft. What is the necessary slope to be given to the

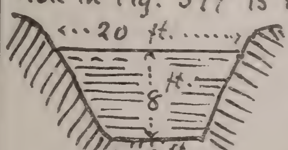


Fig. 597.

bed ? (slope of bed = that of surface, here) Ft.-lb.-sec.

The mean velocity $v = Q \div F = 300 \div \frac{1}{2}(20+8)8 = \left\{\begin{matrix} 2.67 \\ \text{ft. per sec.} \end{matrix}\right.$ (so that the surface velocity of mid-channel in any section would probably be $c_{0\,max} = v \div 0.83 = 3.21$ ft. per sec. (eq. 2 § 492)

The wetted perimeter $w = 8 + \sqrt{8^2+6^2} = 28$ ft. and $\therefore$ the mean hydraulic depth $= R = F \div w = 112 \div 28 = 4$ feet

Using Weisbach's tables for f we find for $v = 2.67$ ft. per sec., $f = .00795$ whence from eq. (3) the fall of the surface (and hence of the bed) for each 1000 ft. of length must be made equal to

$$h = \frac{.00795 \times 1000 \times 28\,(2.67)^2}{112 \times 2 \times 32.2} = 0.221 \text{ feet.}$$

$\left[\therefore \text{the slope } s = \frac{h}{l} = .000221\right]$

Example 2. The desired transverse water-section of a canal is given in Fig. 598. The slope is to be 3 ft. in 1600 i.e. $s = 3 \div 1600$, or for $l = 1600$ ft. $h = 3$ ft. What must be the velocity (mean) of each section, for a uniform motion; the corresponding vol. delivered per sec. $Q = Fv = ?$

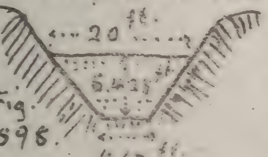


Fig. 598.

Solution. From figure $F = 79.28$ sq. ft. $\therefore R = F \div w = 3.215$ ft.

The velocity v, on which f depends, being unknown we use Eytelwein's value in eq. 3 and solve for v, whence (ft. lb. sec.)

$$v = \sqrt{\frac{2g}{f} \cdot \frac{hR}{l}} = \sqrt{\frac{2 \times 32.2 \times 3 \times 3.215}{.007565 \times 1600}} = 7.16 \text{ ft. p. sec.} \left\{\begin{matrix} 1^{st} \\ \text{approx.} \end{matrix}\right.$$

for which, acc. to Weisbach $f = .00758$, and $\therefore$ as a second approx.^{n} we obtain $v \doteq 7.15$ ft. p. sec. and the vol. of deliv. $Q = Fv = 79.28 \times 7.15 = 567.6 \left\{\begin{matrix} \text{cub.} \\ \text{ft.} \\ \text{p.} \\ \text{sec.} \end{matrix}\right.$

Example 3. If the bed of a creek falls 20 inches every 1500 ft. of length, what vol. of water must be flowing to maintain a uniform mean depth of $4\frac{1}{2}$ feet, the corresponding surface-width being 40 ft. and wetted perimeter 46 ft.? The bed is "in moderately good order and regimen"; use Kutter's formula, putting $n = 0.030$; (ft. and sec.)

First we have $\sqrt{Rs} = \sqrt{(40 \times 4\frac{1}{2}) \div \left(46 \times \frac{1500}{20/12}\right)} = .066$ while $\sqrt{R\,\text{ft.}} = 1.98$ and the slope $= s = \frac{20}{12} \div 1500 = .00111$

$$\therefore \quad v = \frac{\left[41.6 + \frac{1.811}{.030} + \frac{.00281}{.00111}\right] \times 0.066}{1 + \left[41.6 + \frac{.00281}{.00111}\right] \frac{0.030}{1.98}} = \frac{104.42 \times .066}{1.665}$$

$\therefore v = 4.13$ ft. p. sec.

[N.B. Weisbach works this example by eq. (3) and derives $v = 6.1$ ft. per sec., but his values for f are based on experiments with,

comparatively small streams and smooth beds.]

With $v = 4.13$ ft. p. sec. Vol. of disch. $= Q = Fv = 743.4$ cub. ft. p. sec.

495. HYDRAULIC MEAN DEPTH FOR A MINIMUM FRICTIONAL RESISTANCE. We note from eq. (3) § 494 that if an open channel of given length l and sectional area F is to deliver a given volume, Q, per time-unit with *uniform motion*, so that the common mean velocity v of all sections $(= Q \div F)$ is also a given quantity, the necessary fall $= h$, or slope $= s = h \div l$, is seen to be inversely proportional to R the hydraulic mean depth of the section, $= (F \div w) =$ sectional area $\div$ wetted perimeter.

For h to be as small as possible we may design the *form* of transverse section so as to make R as large as possible; i.e. to make the wetted perimeter a minimum for a given F, for in this way a minimum of frictional contact, or area of rubbing surface, is obtained for a prism of water of given sectional area F and given length l.

In a closed pipe running full the wetted perimeter is the *whole* perimeter and if the given sectional area is shaped in the form of a circle the wetted perimeter $= w$, is a minimum (and R a maximum). If the full pipe must have a polygonal shape of n sides then the regular polygon of n sides will provide a minimum w.

Whence it follows that if the pipe or channel is running half-full, and thus becomes an open channel, the semi-circle, of all curvilinear water profiles gives a minimum w or wet. peri. Also of all trapezoidal profiles with banks at 60° with horiz. the half of a regular hexagon gives a minimum w. Among all rectangular sections the half-square gives a minimum w; and of all half-octagons the half of a regular octagon gives a min. w (and max. R) for a given F. See Fig. 599 for all these.

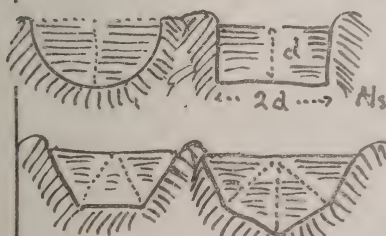


Fig. 599

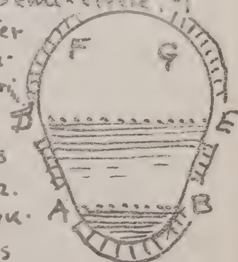


Fig. 600.

The egg-shaped outline, Fig. 600, small end down, is frequently given to sewers in which the maintenance of the same velocity for any depth of flow is important, because the lower portion ABC, providing for the lowest stage AB of flow, being nearly semicircular will induce a velocity of flow (s being fixed) about as great as that occurring when the water

flows at the highest stage DE; the reason being that ABC, from its advantageous form has nearly as great a hyd. mean depth, R, as DEC. That is, $F \div w$ for ABC $= F \div w$ for DEC, nearly.

495a. TRAPEZOID OF FIXED SIDE-SLOPE. For large artificial water courses and canals the trapezoid, or three-sided water-profile, (symmetrical) is customary and the inclination of the bank, or angle θ with the horizontal, Fig. 601, is often determined by the nature of the material composing it, to guard against washouts, caving in, &c. We are ∴ concerned with the following problem:

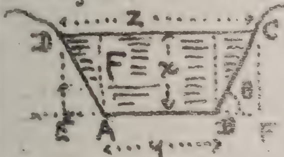


Fig. 601

Given the area, F, of the transv. section and the angle θ, required the value of the depth x (or of upper width z, or of lower width y, both of which are functions of x) to make the hydraulic mean depth, $R = F \div w$, a maximum; or $w \div F$ a minimum; F is constant.

From figure we have $w = \overline{AB} + 2\overline{BC} = y + 2x \operatorname{cosec.}\theta$ .....(1)

and $F = yx + x^2 \cot.\theta$ whence $y = \frac{1}{x}.(F - x^2 \cot.\theta)$ ....(2)

substituting which in (1) and dividing by F, noting that $\operatorname{cosec}\theta - \cot\theta = \frac{2-\cos\theta}{\sin\theta}$

we have $\frac{w}{F} = \frac{1}{R} = \frac{1}{x} + \frac{2-\cos\theta}{F\sin\theta}.x$ .....(3) For a minim. w, we put

$\frac{d(\frac{w}{F})}{dx} = 0$; i.e. $-\frac{1}{x^2} + \frac{2-\cos\theta}{\sin\theta} = 0$ ∴ $x, \left\{\begin{matrix}\text{max. or} \\ \text{for min. } w\end{matrix}\right\} = \pm\sqrt{\frac{F\sin\theta}{2-\cos\theta}}$

The + sign renders the second derivative of $\frac{w}{F}$ positive ∴ for a

$$\left.\begin{matrix}\text{min. } w \\ \text{or max. } R\end{matrix}\right\} x = (\text{call it})\ x' = \frac{\sqrt{F\sin\theta}}{\sqrt{2-\cos\theta}} \text{ ....(4)}$$

while the corresponding values for the other dimensions are:

$y' = \frac{F}{x'} - x'\cot\theta$ ....(5) and $z' = y' + 2x'\cot\theta = \frac{F}{x'} + x'\cot\theta$ ......(6)

For the corresponding hyd. mean depth R' [see (3)], i.e. the max. R,

$\frac{1}{R'} = \frac{1}{x'} + \frac{2-\cos\theta}{F\sin\theta}x' = \frac{2}{x'}$ ...(7) ∴ $R' = \frac{1}{2}x' = \frac{1}{2}\sqrt{\frac{F\sin\theta}{2-\cos\theta}}$ .....(8)

Equations (4)(5).....(8) hold good then, for the trapezoidal section of least frictional resistance for a given angle θ.

Example. Required the dimensions of the trapezoidal section of minimum frictional resistance for $\theta = 45°$, which with six inches fall every 1200 feet is required to deliver 350 cub. ft. of water per minute with uniform motion.

Here we have given, with uniform motion, h, l, and Q; with the requirement that the section shall be trapezoidal with $\theta = 45°$, and of minimum frictional resistance. We have the following equa-

tions to work with. EQ. OF CONTINUITY ... $Q = Fv$ ....(1′)

eq. 8 preceding, for condition of least resistance } .... $R' = \frac{1}{2}\sqrt{\frac{\sin\theta}{2-\cos\theta}}\sqrt{F}$ ...(2′)

From eq. 3 § 494 } for uniform motion $h = f\frac{l}{R'}\cdot\frac{v^2}{2g}$ ...... (3′)

There are three unknown quantities v, F, and R'. Solve (1′) for v; solve (2′) for R'; substitute their values in (3′) whence $h = \frac{f}{2g}\cdot\frac{2l}{\sqrt{F}}\cdot\frac{Q^2}{F^2}$; $\therefore F = \left[\frac{2fl\sqrt{2-\cos\theta}\ Q^2}{2gh\sqrt{\sin\theta}}\right]^{\frac{2}{5}}$ ...(4′)

in which using Eytelwein's value for f ($= .007565$) and above numerical data we have (ft. lb. sec.) } $F = \sqrt[5/2]{\frac{2 \times .00756 \times 1200 \times \sqrt{2-.707} \times 6^2}{2 \times 32.2 \times \frac{1}{2}\sqrt{.707}}}$

$= 3.78$ sq. ft. as a 1st approximation. With this value of F, $v = 6 \div 3.78 = 1.58$ ft. per sec; for this Weisbach gives $f = .0083$ and a second approximation for F gives $F = 3.902$ sq. feet (near enough). Hence we now have from eq. (4); depth $= x' = \frac{\sqrt{3.902 \times .707}}{\sqrt{2 - .707}} = 1.48$ ft.

width at bottom, eq. (5), $y' = \frac{3.78}{1.48} - 1.48 \times 1.00 = 1.226$ feet and

width at top, eq. (6), $z' = 1.226 + 2 \times 1.48 \times 1.00 = 4.184$ ft.

496. VARIABLE MOTION. If a steady flow of water of a delivery $Q = Fv$, = constant, takes place in a straight open channel the slope of whose bed has not the proper value to maintain a uniform motion, then *variable motion* ensues (the flow is still steady, however); i.e., although the mean velocity of any one transverse section remains fixed (with lapse of time) this velocity has different values for different sections, but as the eq. of continuity $Q = Fv = F_1v_1 = F_2v_2$, etc still holds, (since the flow is steady) the different sections have different areas.

Fig. 602

If, Fig. 602, a stream of water flows down an inclined trough without friction, the relation between the velocities v_0 and v_1 at any two sections 0 and 1, will be the same as for a material point sliding down a guide without friction, (see § 79, latter part) viz.

$$\frac{v_1^2}{2g} = \frac{v_0^2}{2g} + h \;\ldots(1)$$ .. an equation of heads (really a case of Bernoulli's theorem § 451) But considering friction on the bed we must subtract the mean friction-head $f\frac{l}{R}\cdot\frac{v_m^2}{2g}$, (see eqs. 3 and 3′ § 494) lost between O and 1; it may also be written $f\frac{l\,w_m}{F_m}\frac{v_m^2}{2g}$

∴ eq.(1) will become $$\frac{v_1^2}{2g} = \frac{v_0^2}{2g} + h - \frac{f\,w_m\,l}{F_m}\,\frac{v_m^2}{2g} \;\ldots\ldots(2)$$

which is the formula for variable motion, in which l is the length of the section considered, which should be taken short enough to consider the surface straight between the end sections and the latter should differ but little. The subscript m may be taken as referring to the section midway between the ends so that $v_m^2 = \frac{1}{2}(v_0^2+v_1^2)$ the wetted perimeter $w_m = \frac{1}{2}(w_0+w_1)$, while $F_m = \frac{1}{2}(F_0+F_1)$ whence

eq.(2) may be written $$h = \frac{v_1^2}{2g} - \frac{v_0^2}{2g} + \frac{\frac{1}{2}fl(w_0+w_1)}{F_0+F_1}\cdot\frac{v_0^2+v_1^2}{2g} \;\ldots\ldots(3)$$

which again may be transformed by putting $v_0 = Q \div F_0$, $v_1 = Q \div F_1$ into

$$h = \left[\frac{1}{F_1^2} - \frac{1}{F_0^2} + \frac{\frac{1}{2}fl(w_0+w_1)}{F_0+F_1}\left(\frac{1}{F_0^2}+\frac{1}{F_1^2}\right)\right]\frac{Q^2}{2g} \;\ldots(4)$$

or, solving for Q,

$$Q = \frac{\sqrt{2gh}}{\sqrt{\frac{1}{F_1^2} - \frac{1}{F_0^2} + \frac{\frac{1}{2}fl(w_0+w_1)}{F_0+F_1}\left[\frac{1}{F_0^2}+\frac{1}{F_1^2}\right]}} \;\ldots(5')$$

From eq.(4), having given the desired shape, areas, etc. of the end-sections and the volume of water Q to be carried per unit of time, we may compute the necessary fall, h, of the surface; while from (5′), having observed in an actual water-course the values of the sectional areas F_0 and F_1, the wetted perimeters w_0 and w_1, the length, l, of the portion considered, we may calculate Q and thus gauge the stream approximately, without making any velocity measurements

Example. (From Weisbach's Mechanics.) The surface of a creek falls 9.6 inches in 300 feet (flow steady) the mean value of the wetted perimeter is $w_m = \frac{1}{2}(w_0+w_1) = 40$ ft.; while $F_0 = 70$ sq.ft. and $F_1 = 60$ sq.ft. With $f = .007565$ we have (eq. 5′) (ft. lb. sec.)

$$Q = \frac{\sqrt{2\times 32.2\times 0.8}}{\sqrt{\frac{1}{60^2} - \frac{1}{70^2} + \frac{.007565\times 300\times 40}{130}\left(\frac{1}{60^2}+\frac{1}{70^2}\right)}}$$

$= 354\frac{1}{2}$ cubic ft. per sec. With this value of Q we have for the value v_m = veloc. of mid. section, $v_m = 354.5 \div 65 = 5.45$ whence $f = .00765$, with which we obtain $Q = 352.5$ cub. ft. p. sec.

However in another reach of the same creek we find such values of F_1, F_0, etc that eq. (5) gives $Q = 365$ cub. ft. p. sec. ∴ the mean of the two is $Q = \frac{1}{2}(365 + 352.5) = 305.$ c. ft. p. sec.

497. BENDS IN AN OPEN CHANNEL. According to Humphreys and Abbot's researches on the Mississippi river the loss of head due to a bend may be put (not homog.) in which v must be in ft. per sec. and

$$h_r = \frac{v^2}{536}\frac{6\delta}{\pi} \quad \ldots\ldots (1)$$

δ, the angle ABC Fig. 603, must be in π-meas, i.e. in radians.

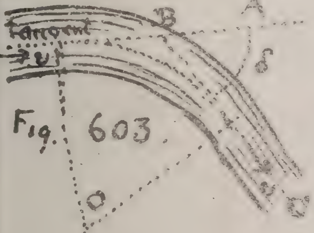


Fig. 603.

The section F must be > 100 sq. ft. and the slope $s < .0008$. v is the mean veloc. of the water. Hence if a bend occurred in a portion of a stream of length l, eq. (3) of §494 becomes

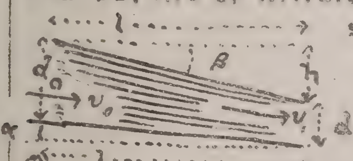
$$h = \frac{fl}{R}\frac{v^2}{2g} + \frac{6}{536}\frac{v^2\delta}{\pi} \quad \ldots \text{ft. \& sec. only} \ldots (2)$$

while eq. (2) of §496 for variable motion would then become

$$\frac{v_1^2}{2g} = \frac{v_0^2}{2g} + h - \frac{f w_m l}{F_m}\frac{v_m^2}{2g} - \frac{6}{536}\frac{v^2\delta}{\pi} \quad \ldots \text{ft. and sec. only} \ldots (3)$$

v and δ as above.

498. EQUATIONS FOR VARIABLE MOTION INTRODUCING THE DEPTHS OF WATER. Fig. 604. The slope of the bed being $\sin\alpha$ (or simply α π-meas.) while that of the surface is different viz.



$\sin\beta = s = h \div l$, we may write

$$h = d_0 + l\sin\alpha - d_1,$$

in which d_0 and d_1 are the depths at the end sections of the portion considered (steady flow with variable motion) This gives, in eq. 4 §496, solving for l,

$$\therefore l = \frac{d_0 - d_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}}{\frac{f w_m}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g}} \quad \ldots\ldots (1)$$

From which, knowing the slope of the bed and the shape and size of the end-sections, also the discharge Q, we may compute the length or distance, l, between two sections whose depths differ by an assigned amount $(d_0 - d_1)$. But we cannot compute the change of depth for an assigned length l from (6). However, if the mean width b of the stream is constant eq. 6 may be much simplified by introducing some approximations as follows: We may put

$$\left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \frac{F_0^2 - F_1^2}{F_1^2 F_0^2} \cdot \frac{Q^2}{2g} = \frac{(F_0 - F_1)(F_0 + F_1)}{F_1^2} \cdot \frac{v_0^2}{2g}$$

$$= \frac{(d_0 - d_1)(d_0 + d_1)}{d_1^2} \cdot \frac{v_0^2}{2g}, \text{ which approx.} = \frac{2(d_0 - d_1)}{d_0} \cdot \frac{v_0^2}{2g}\text{; and, similarly,}$$

$$\frac{w_m}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} = \frac{w_m(F_0^2 + F_1^2)}{(F_0 + F_1)F_1^2} \cdot \frac{v_0^2}{2g} = (\text{approx.}) = \frac{w_m}{d_0 b} \frac{v_0^2}{2g}$$

Hence by substitution in eq. 6 we have, approx.,

$$l = \frac{(d_0 - d_1)\left[1 - \frac{2}{d_0} \cdot \frac{v_0^2}{2g}\right]}{f \frac{w_m}{d_0 b} \frac{v_0^2}{2g} - \sin\alpha} \quad \ldots (7.)$$

CHAP. VI.

DYNAMICS OF GASEOUS FLUIDS.

499. STEADY FLOW OF A GAS. [N.B. The student should now review §451 up to eq. (5')] The differential equation from which Bernoulli's Theorem was derived for any liquid, <u>without friction</u>, was

(eq. 5' §451) $$\frac{1}{g} v\,dv + dz + \frac{1}{\gamma} dp = 0 \quad \ldots\ldots (A)$$

is equally applicable to the steady flow of a gaseous fluid, but with this difference in subsequent work that the *heaviness*, γ (§7), of the gas passing different sections of the pipe or stream line is or may be different (though always the same at a given point or section, since the flow is steady) For the present we neglect friction and consider the flow from a large receiver, where the great body of the gas is practically at rest, through an orifice in a thin plate or a short nozzle with a rounded entrance.

In steady flow of a gas, since γ is different at different points the EQ. OF CONTINUITY takes the form ...... WEIGHT per time unit $\} = F_1 v_1 \gamma_1 = F_2 v_2 \gamma_2 = \text{etc.} \ldots (a)$

i.e. the <u>weight</u> of gas passing any section, of area F, per unit of time, is the same as for any other section, or $Fv\gamma =$ constant, γ being the heaviness at the section, and v the velocity.

500. FLOW THROUGH AN ORIFICE. REMARKS. In Fig. 605 we have a large rigid receiver containing gas at some tension p_n higher than that p_m of the still outside air (or gas), and some absolute temperature T_n, and of some heaviness γ_n; that is in a *state n*. The small orifice of area F being opened the gas begins to escape, and if the receiver is very large, or if the supply continually kept up (by a blowing engine, e.g.) after a very short time the flow becomes steady.

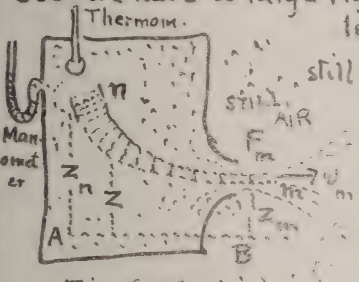


Fig. 605.

Let nm represent any stream-line (§ 454) of the flow. According to the ideal subdivision of this stream line into laminae of equal mass or weight (not volume, necessarily) in establishing eq. A for any one lamina, each lamina in the lapse of time dt moves into the position just vacated by the lamina next in front and assumes precisely the same velocity, and volume (and ∴ heaviness), as that front one had at the beginning of the dt. In its progress toward the orifice it expands in volume, its tension diminishes, while its velocity, insensible at n, is gradually accelerated on account of the pressure from behind always being greater than that in front, until at m, in the "*throat*" of the jet, the velocity has become v_m, the pressure (i.e. tension) has fallen to a value p_m, and the heaviness has changed to γ_m. The temperature T_m (absol.) is less than T_n, since the expansion has been rapid, and does not depend on the temperature of the outside air or gas into which efflux takes place, though, of course, after the effluent gas is once free from the orifice it may change its temperature in time.

We assume the pressure p_m (in throat of jet) to be equal to that of the outside medium (as was done with flow of water) so long as that outside tension is $> .527\, p_n$, but if it is $< .527\, p_n$ and is even zero (a vacuum), experiment seems to show

that p_m remains equal to 0.527 of the interior tension p_n, probably on account of the expansion of the effluent gas beyond the throat at B Fig. 606, so that although the tension in the outer edge, at a, of the jet is equal to that of the outside medium, the tension at m is greater because of the centripetal and centrifugal forces developed by the curved filaments between a and m. (See § 298)

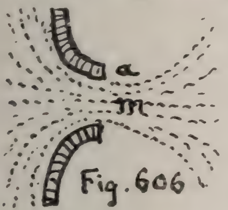


Fig. 606

501. FLOW THROUGH AN ORIFICE; HEAVINESS ASSUMED CONSTANT DURING FLOW; THE WATER FORMULA.

If the inner tension p_n exceeds the outer, p_m, but slightly, we may assume that, like water, the gas remains of the same heaviness during flow. Then, for the simultaneous advance made by all the laminae of a stream line, Fig. 605, in the time dt, we may conceive an equation like eq. A written out for each lamina between n and m, and corresponding terms added; i.e.

$$\begin{matrix}\text{(GENERAL)}\\ \text{(FOR ORIFICES)}\end{matrix} \ldots \frac{1}{g}\int_n^m v\,dv + \int_n^m dz + \int_n^m \frac{dp}{\gamma} = 0 \ldots\ldots \quad (B)$$

In general γ is different in the different laminae, but in the present case, it is assumed the same in all, hence, with m as datum-level and h = vertical distance from n to m, we have from eq. B

$$\frac{v_m^2}{2g} - \frac{v_n^2}{2g} + 0 - h + \frac{p_m}{\gamma} - \frac{p_n}{\gamma} = 0 \ldots\ldots (1.)$$

But we may put $v_n = 0$, while h, even if several feet, is small compared $\frac{p_m}{\gamma} - \frac{p_n}{\gamma}$. [E.g., with $p_m = 16$ lbs. per sq. in, and $p_n = 15$, we have $\frac{p_m}{\gamma} - \frac{p_n}{\gamma}$, for air at freezing temp, = 1638 feet; see eq. δ § 441]

$$\left.\begin{matrix}\therefore \text{ eq.(1)}\\ \text{reduces to}\end{matrix}\right\} \ldots \frac{v_m^2}{2g} = \frac{p_n - p_m}{\gamma_n} \ldots\ldots \left\{\begin{matrix}\text{WATER FORMULA;}\\ \text{for small difference}\\ \text{of pressures, only}\end{matrix}\right\} \ldots (2)$$

The interior absol. temp. T_n being known, the γ_n (interior heaviness) may be obtained from § 437, viz. from $\gamma_n = p_n \gamma_0 T_0 \div T_n p_0$ and the volume then obtained, per unit of time (first solving (2) for v_m) viz:

$$Q_m = F_m v_m \ldots\ldots\ldots\ldots (3)$$

where F_m is the sectional area of the jet at m. If the mouth

piece or orifice has well rounded interior edges, its sectional area F may be taken as the area F_m. But if it is an orifice in "thin plate", putting coefficient of contraction $= C = 0.60$ we have $F_m = CF = 0.60F$ $\therefore$ $Q_m = 0.60 F v_m$ ......(4)

This volume Q_m is that occupied by the flow per time unit when in state m, and we have assumed that $\gamma_m = \gamma_n$; $\therefore$ the weight of flow per time-unit is

$$G = Q_m \gamma_m = F_m v_m \gamma_m = F_m v_m \gamma_n \text{ ......} (5)$$

Example. In the testing of a blowing engine it is found capable of maintaining a pressure of 18 lbs. per sq. inch in a large receiver, from whose side a blast is steadily escaping thro' a "thin plate" orifice (circular) having an area $F = 4$ sq. inches. The interior temperature is 20° Cent. and the outside tension 15 lbs. per sq. in.

Required the discharge of air per second, both volume and weight. The data are: $p_n = 18$ lbs. per sq. in. $T_n = 293°$ Abs. Cent. $F = 4$ sq. inches and $p_m = 15$ " " " " Use ft. lb. sec.

First, the heav. in the receiver $\Big\}$ is $\gamma_n = \dfrac{p_n}{p_0}\cdot\dfrac{T_0}{T_n}\gamma_0 = \dfrac{18}{14.7}\cdot\dfrac{273}{293}\times .0807 = .089 \left\{\begin{matrix}\text{lbs.}\\ \text{p.}\\ \text{cub.}\\ \text{ft.}\end{matrix}\right.$

From eq. (2)

$$v_m = \sqrt{2g\frac{p_n - p_m}{\gamma_n}} = \sqrt{\frac{2\times 32.2\,[144\times 18 - 144\times 15]}{0.089}} = 555.3 \text{ ft. p sec.}$$

[97% of this would be more correct on account of friction]

$\therefore Q_m = F_m v_m = .6 F v_m = \dfrac{6}{10}\cdot\dfrac{4}{144}\times 555.3 = 9.24$ cub. ft. per sec.

at tension 15 lbs. per sq. in., and of heaviness (by hypothesis) $=$ 0.89 lbs. per cub. ft. $\therefore$ weight $= G = 9.24 \times .89 = .82$ lbs. per sec.

The theoretical power of the air-compressor or blowing engine to maintain this steady flow can be computed as in Ex. 3 § 447.

502. FLOW THROUGH AN ORIFICE ON THE BASIS OF MARIOTTE'S LAW. Since the gas really expands during flow thro' an orifice, and $\therefore$ changes its heaviness, Fig. 605, we approximate more nearly to the truth in assuming this change of density to follow Mariotte's law, i.e. that the heaviness varies directly as the pressure, and thus imply that the temperature remains unchanged during the flow. We again integrate the terms of

eq. B, but take care to note that, now, γ is variable (i.e. different in different laminae at the same instant) and ∴ express it in terms of the variable p (from eq. 2 § 440) thus: $\gamma = (\gamma_n \div p_n) p$

∴ the term $\int_n^m \frac{dp}{\gamma}$ of eq. B becomes $\frac{p_n}{\gamma_n} \int_n^m \frac{dp}{p} = \frac{p_n}{\gamma_n} \log_\varepsilon \frac{p_n}{p_m}$ ....(1)

∴, integrating all the terms of eq. B, neglecting h and calling v_n zero we have

$$\frac{v_m^2}{2g} = \frac{p_n}{\gamma_n} \log_\varepsilon \frac{p_n}{p_m} \;\cdots\cdots \left\{\begin{array}{l}\text{EFFLUX BY} \\ \text{MAR.S LAW}\end{array}\right\} \cdots (2)$$

thro' ORIFICE

As before $\gamma_n = \frac{T_0}{T_n} \cdot \frac{p_n}{p_0} \gamma_0$; and flow of volume per time unit at m ...... $= Q_m = F_m v_m$ ...(3)

while if the orifice is in thin plate, F_m may be put $= .60\,F$; and the WEIGHT OF THE FLOW PER TIME-UNIT $= G = F_m v_m \gamma_m$ ...... (4)

If mouth-piece is rounded $F_m = F$ = area of exit orifice of mouth-piece

Example. Applying eq. (2) to the data of the example in § 501, where γ_n was found to be .089 lbs. per cubic ft., we have [ft., lb., sec.]

$$v_m = \sqrt{2g \frac{p_n}{\gamma_n} \log_\varepsilon \frac{p_n}{p_m}} = \sqrt{2 \times 32.2 \times \frac{18 \times 144}{.089} \times 2.3025 \times \log_{10}\left[\frac{18}{15}\right]} = 584.7 \text{ ft. p. sec.}$$

∴ $Q_m = F_m v_m = 0.60 \times \frac{4}{144} \times 584.7 = 9.745$ cub. ft. per sec. Since the heaviness at m is, from Mar.s Law, $\gamma_m = \frac{p_m}{p_n} \gamma_n = \frac{15}{18}$ of .089, i.e., $\gamma_m = .0741$ lbs. per cub. foot, ∴ the weight of the discharge is $G = Q_m \gamma_m = 9.745 \times .0741 = 0.722$ lbs. per sec., or about 12 per cent. less than that given by the "water formula". If the difference between the inner and outer tensions had been less, the discrepancy between the results of the two methods would have been less.

503. ADIABATIC EFFLUX FROM AN ORIFICE. It is most logical to assume that the expansion of the gas approaching the orifice, being rapid, is *adiabatic* (§ 442). Hence (especially when the difference between the inner and outer tensions is considerable) it is more accurate to γ as varying acc. to POISSON'S LAW, eq. (1) § 442, i.e. $\gamma = [\gamma_n \div p_n^{2/3}]\, p^{2/3}$, in integrating eq. B. Then the term $\int_n^m \frac{dp}{\gamma}$ will $= \frac{p_n^{2/3}}{\gamma_n} \int_n^m p^{-\frac{2}{3}} dp = \frac{3 p_n^{2/3}}{\gamma_n} \left[p_m^{1/3} - p_n^{1/3}\right]$

$= -\frac{3 p_n}{\gamma_n}\left[1 - \left(\frac{p_m}{p_n}\right)^{1/3}\right]$; and eq. B, neglecting h as before,

and with $v_n = 0$ becomes, Fig. 605 $\Bigg\}$ $$\frac{v_m^2}{2g} = \frac{3p_n}{\gamma_n}\left[1-\left(\frac{p_m}{p_n}\right)^{1/3}\right] \cdots \left\{\begin{array}{l}\text{ADIAB.}\\ \text{FLOW;}\\ \text{ORIFICE}\end{array}\right\} \cdots\cdots (1.)$$

Having observed p_n and T_n in reservoir we compute $\gamma_n = \frac{p_n \gamma_0 T_0}{T_n p_0}$ (from § 437). The gas at m, just leaving the orifice, having expanded adiabatically from the state n to the state m, has cooled to a temp. T_m (absol.) found thus (§ 442) $T_m = T_n\left(\frac{p_m}{p_n}\right)^{1/3} \cdots\cdots (2)$ and a heaviness $\Big\}$ $\gamma_m = \gamma_n\left(\frac{p_m}{p_n}\right)^{2/3} \cdots (3)$

and the flow per second immediately on exit occupies a volume $Q_m = F_m v_m \cdots\cdots (4)$ and weighs $G = F_m v_m \gamma_m \cdots\cdots (5)$

Example 1. Let the interior conditions in the large reservoir of Fig. 605 be as follows (*state n*): $p_n = 22\frac{1}{2}$ lbs. per sq. in. and $T_n = 294°$ Abs. Cent. (i.e. 21° Cent.); while externally the tension is 15 lbs. per sq. inch, which may be taken as $= p_m =$ tension at m, the *throat* of jet. The opening is a circular orifice in "*thin plate*" and of one inch diameter. Required the weight of the discharge per second [Ft. lb. sec.; $g = 32.2$]

First, $\gamma_n = \frac{22.5 \times 144}{15 \times 144} \cdot \frac{273}{294} \times .0807 = 0.114$ lbs. per cubic foot.

Then, (1),
$$v_m = \sqrt{2g\,\frac{3p_n}{\gamma_n}\left[1-\left(\frac{p_m}{p_n}\right)^{1/3}\right]} = \sqrt{\frac{2 \times 32.2 \times 3 \times 22.5 \times 144}{0.114}\left[1-\sqrt[3]{\tfrac{2}{3}}\right]}$$

$= 844.$ ft. per sec. Now $F = \frac{1}{4}\pi\left(\frac{1}{12}\right)^2 = .00546$ sq. ft.

$\therefore Q_m = CF\,v_m = .60F\,v_m = 0.60 \times .00546 \times 844 = 2.765$ $\left\{\begin{array}{l}\text{cub.}\\ \text{ft.}\\ \text{p.}\\ \text{sec.}\end{array}\right.$

at a temp. of $T_m = 294\sqrt[3]{\frac{2}{3}} = 257°$ abs. Cent. $= -16°$ Cent.

and of a heaviness $\gamma_m = 0.114\sqrt[3]{\left(\frac{2}{3}\right)^2} = 0.085$ lbs. per cub. ft

So that the weight of flow p. sec $= G = Q_m\gamma_m = 2.765 \times .085 = .235$ $\left\{\begin{array}{l}\text{lbs.}\\ \text{per}\\ \text{sec.}\end{array}\right.$

Example 2. Let us treat the same example already solved by the two preceding approx. methods (§§ 501 and 502) by the present more accurate equation of adiabatic flow, eq. (1).

The data were: (Fig. 605) $\Big\{$ $p_n = 18$ lbs. per sq. in ; $T_n = 293°$ Abs. Cent. ; $p_m = 15$ " " " " ; and $F = 4$ sq. inches

[F being the area of orifice]. γ_n was found $= .089$ lbs. per cub. ft. in § 501. ∴ from eq.(1)

$$v_m = \sqrt{\frac{2 \times 32.2 \times 3 \times 18 \times 144}{.089}\left[1 - \sqrt[3]{\frac{5}{6}}\right]} = 576.2 \left\{ \begin{array}{l} \text{ft} \\ \text{per} \\ \text{second} \end{array} \right.$$

From (4) $Q_m = F_m v_m = .6 F v_m = .6 \times \frac{4}{144} \times 576.2 = 9.603$ cub. ft. per sec. and since at m it is of a heaviness $\gamma_m = .089 \sqrt[3]{(\frac{15}{18})^2} = .0788$ lbs. per cub. foot we have

WEIGHT OF FLOW PER SEC. $= G = Q_m \gamma_m = 9.603 \times .0788 = 0.756$ lbs. per sec.

Comparing the three methods for this problem

By the "*water formula*" $G =$ 0.82 lbs. per sec.
" " *Mariotte's law form.* $G =$ 0.722 " " ".
" " *Adiabatic formula* $G =$ .756 " " ".

504. PRACTICAL NOTES. THEORETICAL MAX. FLOW OF WEIGHT.

If in the equations of § 503 we write for brevity $p_m \div p_n = x$ we derive by substitution from (1) and (3) in (5)

$$\left.\begin{array}{l}\text{WEIGHT OF FLOW}\\ \text{PER UNIT OF TIME}\end{array}\right\} = G = Q_m \gamma_m = F_m \sqrt{2 g p_n \gamma_n \left[1 - x^{1/3}\right]}\, x^{2/3} \quad \ldots\ldots (1)$$

This function of x, considered variable, is of such a form as to be a maximum for

$$x = (p_m \div p_n) = \left(\frac{4}{5}\right)^3 = .512 \quad \ldots\ldots (2)$$

i.e., theoretically, if the state n inside the reservoir remains the same, while the outside tension (considered $= p_m$ of jet, Fig. 605) is made to assume lower and lower values (and hence $x = p_m \div p_n$ to diminish in the same ratio the maximum flow of weight per unit of time will occur when $p_m = .510\, p_n$ a little more than half the inside tension. (With the more accurate value 1.41 (1.408), instead of $3/2$, see § 442, we would obtain .527 instead of .512 for dry air; see § 500)

Prof. Cotterill says (p. 544 of his "Appl. Mechanics") "The diminution of the theoret. discharge on diminution of the external pressure below the limit just now given, is an anomaly which had always been considered as requiring explanation, and M. St. Venant had already suggested that it could not actually occur. In 1866 Mr. R. D. Napier showed by experiment that the weight of steam of given pressure discharged from an orifice really is independent of the pressure of the medium into which efflux takes place;

and in 1872 Mr. Wilson confirmed this result by experiments on the reaction of steam issuing from an orifice."

"The explanation lies in the fact that the pressure in the centre of the contracted jet is not the same as that of the surrounding medium. The jet after passing the contracted section suddenly expands, and the change of direction of the fluid particles gives rise to centrifugal forces" which cause the pressures to be greater in the centre of the contracted section than at the circumference; see Fig. 606.

Prof. Cotterill then advises the assumption that $p_m = .527 p_n$ (for air and perfect gases) as the mean tension in the jet at m (Fig. 606), whenever the outside medium is at a tension less than $.527 p_n$. He also says "Contraction and friction must be allowed for by the use of a co-efficient of discharge the value of which however is more variable than that of the corresponding co-efficient for an incompressible fluid. Little is certainly known on this point." See §§ 505 and 500.

For air the velocity of this max. flow of weight is

$$\text{Vel. of max. } G = \left[997\sqrt{\frac{T_n}{T_0}}\right] \text{ ft. per sec.} \quad \ldots\ldots 3$$

where T_n = Abs temp. in reservoir, and T_0 = that of freezing-point. Rankine's Applied Mechanics (p. 584) mentions experiments of Drs. Joule and Thompson, in which the circular orifices were in a thin-plate of copper and of diameters 0.029 in., 0.053 in., and 0.084 inches, while the outside tension was about one half of that inside. The results were 84 per cent. of those demanded by theory, a discrepancy due mainly, as Rankine says, to the fact that the actual area of the orifice was used in computation instead of the contracted section, i.e. contraction was neglected.

505. CO-EFFICIENTS OF EFFLUX BY EXPERIMENT FOR ORIFICES AND SHORT PIPES. SMALL DIFFERENCE OF TENSIONS. Since the discharge thro' an orifice or short pipe from a reservoir is affected not only by contraction but by slight friction at the edges, even with a rounded entrance, the theoretical results for the volume and weight of flow per unit of time in preceding §§ should be multiplied both by

a coefficient of velocity ϕ and one for contraction C, as in the case of water; i.e. by a coefficient of efflux $\mu, = \phi C$. (Of course, when there is no contraction, $C = 1.00$ and then $\mu = \phi$ as with a well rounded mouth-piece for instance, Fig. 530, and with short pipes)

Hence for practical results, with orifices and short pipes, we should write

$$\left.\begin{array}{l}\text{WEIGHT OF}\\ \text{FLOW PER}\\ \text{TIME-UNIT}\end{array}\right\} = G = \mu F v_m \gamma_m = \mu F \left(\frac{p_m}{p_n}\right)^{2/3} \sqrt{2g\,3\,p_n\gamma_n\left[1-\left(\frac{p_m}{p_n}\right)^{1/3}\right]} \quad \ldots (1)$$

(from the equations of § 503 for adiabatic flow, as most accurate; $p_m : p_n$ may range from $\frac{1}{2}$ to 1.00). F = area of orifice or of discharging end of mouth-piece or short pipe. γ_n = heaviness of air in reservoir and $= T_0 p_n \gamma_0 \div T_n p_0$, eq. 13 of § 437; and μ = the experimental co-efficient of efflux.

From his own experiments and those of Koch, D'Aubuisson and others, Weisbach recommends the following mean values of μ for various mouthpieces, when p_n is not more than $\frac{1}{6}$ larger than p_m (i.e. about 17% larger), for use in eq. (1).

1st. For an orifice in a thin plate .... $\mu = 0.56$

2ndly For a short cylindrical pipe (inner corners not rounded) $\mu = 0.75$

3rdly. For a well rounded mouthpiece (like Fig. 530) .... $\mu = 0.98$

4thly For a short conical convergent pipe (angle about 6°) .... $\mu = 0.92$

Example. (Data from Weisbach's Mech.) "If the sum of the areas of two conical tuyères of a blowing machine is $F = 3$ sq. inches, the temperature in the réservoir 15° Cent., the height of the attached (open) mercury manometer (see Fig. 464) 3 inches, and the height of the barometer in the external air 29 inches" we have (ft. lb. sec.)

$\frac{p_m}{p_n} = \frac{29}{29+3} = \frac{29}{32}$; $T_n = 288°$ Abs. Cent.; $p_n = \left(\frac{32}{30}\right) 14.7 \times 144$ lbs. per sq. ft.; $\gamma_n = \frac{273}{288} \cdot \frac{32}{30} \times .0807 = 0.0816$ lbs. per cub. ft. while $F = \frac{3}{144}$ sq. ft. and (above) $\mu = 0.92$ ∴

$$G = 0.92 \times \frac{3}{144}\left(\frac{29}{32}\right)^{2/3} \sqrt{2 \times 32.2 \times 3 \times \frac{32}{30} \times 14.7 \times 144 \times .0816\left[1-\sqrt[3]{\frac{29}{32}}\right]}.$$

i.e. $G = .6076$ lbs. per second; which will occupy a volume $V_0 = G \div \gamma_0 = G \div .0807 = 7.59$ cub. ft at one atmos. tension and freezing point temp.; while at a temp. of $T_n = 288°$ Abs Cent. and tension of $p_m = \frac{29}{30}$ of one atmos. (i.e. in the state in which it was on entering the blowing engine) it occupied a volume $V = \frac{288}{273} \cdot \frac{30}{29} \times 7.59 = 8.24$ cub. ft.

(The latter is Weisbach's result obtained by an approx. formula)

506. CO-EFS. OF EFFLUX FOR ORIFICES AND SHORT PIPES FOR LARGE DIFFERENCE OF TENSION. For values $> 7/6$ and < 2, of the ratio $p_n : p_m$, of internal to external tension, Weisbach's experiments with *circular orifices in thin plate*, of diameters (= d) from 0.4 inches to 0.8 inches gave the following

$p_n : p_m =$	1.05	1.09	1.40	1.65	1.90	2.00
For $d = .4^{in}$; $\mu =$	.55	.59	.69	.72	.76	.78
" $d = .8$; $\mu =$	.56	.57	.64	.68		.72

whence it appears that μ increases somewhat with the ratio of p_n to p_m, and decreases slightly for increasing size of orifice.

With short cylindrical pipes, internal edges not rounded, and 3 times as long as wide, Weisbach obtained μ as follows:

$p_n : p_m =$	1.05	1.10	1.30	1.40	1.70	1.74
diam. $= .4^{in}$ $\mu =$	.73	.77	.83			
" $= .6^{in}$ $\mu =$				.81	.82	
" $= 1.0^{in}$ $\mu =$						.83

When the inner edges of the 0.4 in. pipe were slightly rounded, μ was found $= 0.93$; while a well rounded mouth-piece of the form shown in Fig. 530 gave a value $\mu =$ from .965 to .968, for $p_n : p_m$ ranging from 1.25 to 2.00.

507. TO FIND THE DISCHARGE WHEN THE INTERNAL PRESSURE IS MEASURED IN A SMALL RESERVOIR OR PIPE, NOT MUCH LARGER THAN THE ORIFICE. Fig. 607.

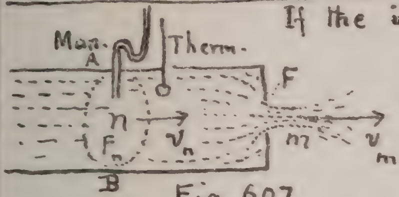


Fig. 607

If the internal pressure p_n, and Temp. T_n must be measured, at n, in a small reservoir or pipe whose sectional area F_n is not very large compared with that of the orifice, F, (or of the jet F_m) the velocity v_n, at n, (vel. of approach) cannot be put = zero. Hence in applying eq. B, § 501, to the successive laminae between n and m, and integrating, we shall have for adiabatic steady flow,

$$\frac{v_m^2}{2g} - \frac{v_n^2}{2g} = \frac{3\,p_n}{\gamma_n}\left[1 - \left(\frac{p_m}{p_n}\right)^{1/3}\right] \text{ .... } (1)$$

instead of eq.(1) of § 503. But from the EQ. OF CONTINUITY for steady flow of gases [eq.(a) of § 499], $F_n v_n \gamma_n = F_m v_m \gamma_m \therefore v_n^2 = \frac{F_m^2 \gamma_m^2}{F_n^2 \gamma_n^2} v_m^2$ while for adiabatic change from n to m, $\gamma_m / \gamma_n = \left(\frac{p_m}{p_n}\right)^{2/3}$; whence by substitution in (1), solving for v_m, we have

$$v_m = \left[\sqrt{2g\,\frac{3\,p_n}{\gamma_n}\left(1 - \left(\frac{p_m}{p_n}\right)^{1/3}\right)}\right] \div \left[\sqrt{1 - \left(\frac{F_m}{F_n}\right)^2 \left(\frac{p_m}{p_n}\right)^{4/3}}\right] \text{ ... } (2)$$

As before

$$\gamma_n = \frac{p_n}{p_0} \cdot \frac{T_0}{T_n}\,\gamma_0 \text{ .... } (3) \; (\S\,437); \text{ and } \gamma_m = \left(\frac{p_m}{p_n}\right)^{2/3}\gamma_n \text{ .... } (4) \; (\S\,445)$$

Having, then, observed p_n, p_m, and T_n, and knowing the area F of the orifice, we may compute γ_n and γ_m and v_m and finally the

WEIGHT OF FLOW PER TIME-UNIT $= G = \mu F v_m \gamma_m$ ....(5)

taking μ from § 505 or 506. In eq.(2) it must be remembered that for an orifice in "thin plate", F_m is the sectional area of the contracted vein, and $= CF$; where C may be put $= \frac{\mu}{.97}$

Example. If the diam. of AB, Fig. 607, is $3\frac{1}{2}$ inches, and that of the orifice, well rounded, = 2 in.; if $p_n = \frac{13}{12}$ atmos. $= \frac{13}{12} \times 14.7 \times 144$ lbs. per sq. ft, while $p_m = \frac{11}{12}$ of an atmos. and $T_n = 283°$ Ab. Cent. so that $p_m / p_n = \frac{11}{13}$; required the discharge per second, using the ft. lb. sec.

From eq.(3).... $\gamma_n = \frac{13}{12} \cdot \frac{273}{283} \times 0.0807 = .08433$ lbs. per cub. ft.

whence (eq.(4)), $\gamma_m = \left(\frac{11}{13}\right)^{2/3} \gamma_n = .07544$ .... " " " "

Then from eq.(2)

$$v_m = \left[\sqrt{\frac{64.4\times 3\times 15.925\times 144}{.08433}\left(1-\left(\frac{11}{13}\right)^{\frac{1}{3}}\right)}\right] \div \left[\sqrt{1-\left(\frac{16}{49}\right)^2\left(\frac{11}{13}\right)^{\frac{4}{3}}}\right]$$

$= 558.1$ ft. per sec. $\therefore G = 0.98\frac{\pi}{4}\left(\frac{1}{6}\right)^2 558.1\times.07544 = \begin{cases}.9003 \\ \text{lbs. p sec.}\end{cases}$

508. TRANSMISSION OF COMPRESSED AIR; THRO' VERY LONG LEVEL PIPES. STEADY FLOW.

Case I. When the difference between the tensions in the reservoirs at the ends of the pipe is small. Fig. 608. Under these circumstances it is simpler to employ the form of formula that

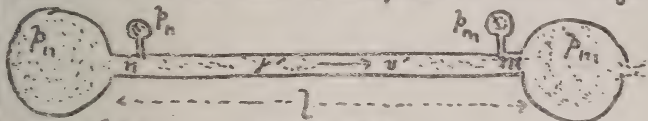


Fig. 608 LEVEL PIPE.

would be obtained for a liquid by applying Bernoulli's Theorem, taking into account the "loss of head" occasioned by the friction on the sides of the pipe. Since the pipe is very long, and the change of pressure small, the mean velocity in the pipe, $= v'$, assumed to be nearly the same at all points along the pipe, will not be large; hence the difference between the veloc. heads at n and m will be neglected; a certain mean heaviness γ' will be assigned to all the gas in the pipe, as if a liquid.

Applying, then, Bernoulli's Theorem, with friction, §474, to the ends of the pipe, n and m, we have

$$\frac{v_m^2}{2g}+\frac{p_m}{\gamma'}+0 = \frac{v_n^2}{2g}+\frac{p_n}{\gamma'}+0-4f\frac{l}{d}\frac{v'^2}{2g} \quad\text{.......(1)}$$

Putting (as above mentioned) $v_m^2 - v_n^2 = 0$, we have, more simply,

$$\frac{p_n-p_m}{\gamma'} = 4f\frac{l}{d}\cdot\frac{v'^2}{2g} \quad\text{..............(2)}$$

The value of f as co-efficient of friction for air in long pipes is found to be somewhat smaller than for water; see next §

509. TRANSMISSION OF COMPRESSED AIR. EXPERIMENTS IN THE ST. GOTHARD TUNNEL, 1878. [See p. 95 of Vol. 24, (Feb '81), Van Nostrand's Engineering Magazine] In these experiments, the temperature and pressure of the flowing gas (air) were observed at each end of a long portion of the pipe which delivered the compressed air to the boring-machines three

miles distant from the tunnel's mouth. The portion considered was selected at a distance from the entrance of the tunnel to eliminate the fluctuating influence of the weather on the temperature of the flowing air. A steady flow being secured by proper regulation of the compressors and distributing tubes, observations were made of the internal pressure (p), internal temp (T), as well as the external, at each end of the portion of pipe considered and also at intermediate points; also of the weight of flow per second $G = Q_0 \gamma_0$ measured at the compressors under standard conditions (0° Cent. and one atmos. tension). Then knowing the p and T at any section of the pipe, the heaviness γ of the air passing that section can be computed [from $\frac{\gamma}{\gamma_0} = \frac{p}{p_0} \cdot \frac{T_0}{T}$] and the veloc. $v = G \div F\gamma$, F being the sectional area at that point. Hence the mean velocity v', and the mean heaviness γ', can be computed for this portion of the pipe whose diam. $= d$ and length $= l$. In the experiments cited it was found that at points not too near the tunnel-mouth the temp. inside the pipe was always about 3° Cent. lower than that of the tunnel. The values of f in the different experiments were then computed from eq. (2) of last § i.e. $\frac{p_n - p_m}{\gamma'} = 4f \frac{l}{d} \cdot \frac{v'^2}{2g}$ ...(2) all the other quantities having been either directly observed, or computed from observed quantities.

THE ST. GOTHARD EXPERIMENTS. (Concrete quant.s reduc.d to English units)

No	l feet	d ft.	γ' lbs. cub. ft	Atmospheres p_n	p_m	$p_n - p_m$ lbs. sq. in.	v' ft. p. sec.	mean temp. C.	Value of f.
1	15092	2/3	0.4058	5.60	5.24	5.29	17.32	21°	.0035
2	15092	2/3	0.3209	4.35	4.13	3.23	16.30	21°	.0038
3	15092	2/3	.2803	3.84	3.65	2.79	15.55	21°	.0041
4	1712	1/2	.3765	5.24	5.00	3.52	37.13	26.5	.0045
5	1712	1/2	.3009	4.13	4.06	1.03	30.82	26.5	.0024 (?)
6	1712	1/2	.2641	3.65	3.54	1.54	29.34	26.5	.0048

In the article referred (Van Nostrand's Mag.) f is not computed. The writer contents himself with showing that Weisbach's values (based on experiments with small pipes and high velocities) are much too great for the pipes in use in the tunnel.

With small tubes an inch or less in diameter Weisbach found, for a veloc. of about 80 ft. per second, $f = .0060$; for still higher velocities f was smaller, approximately

in accordance with the relation $f = .0542 \div \sqrt{v}$ in ft.-p.-sec.
On p. 370, Vol. XXIV, Van Nostrand's Mag., Prof. Robinson of Ohio mentions other experiments with large long pipes. From the St. Gothard experiments a value of $f = .004$ may be inferred for approx. use with pipes from 3 to 8 in. diam.

Example. It is required to transmit, in steady flow, a supply of $G = 6.456$ lbs. of atmospheric air per second thro' a pipe 30 000 ft. in length (nearly six miles) from a reservoir where the tension is 6.0 atmos. to another where it is 5.8 atmos., the mean temp. in the pipe being 80° Fahr. = 24° Cent. (i.e. = 297° Abs. Cent.) Required the proper diameter of pipe; $d = ?$ The value $f = .00425$ may be used.

FT. LB. SEC. The mean volume passing per sec. in the pipe is

$$Q' = G \div \gamma' \quad \text{.......... (3)}$$

The mean velocity may thus written ...... $v = \dfrac{Q'}{F} = \dfrac{Q'}{\frac{1}{4}\pi d^2}$ ...... (4)

The mean density of the flowing air, computed for a mean tens. of 5.9 atmos. (§ 437) is

$$\gamma' = \frac{5.9 \times 14.7}{1 \times 14.7} \cdot \frac{273}{297} \times .0807 = 0.431 \left\{\begin{array}{l}\text{lbs. per}\\ \text{cub. ft.}\end{array}\right.$$

and hence see eq. (3) $\Big\}$ $Q' = \dfrac{G}{\gamma'} = \dfrac{6.456}{0.431} = 14.74$ cub. ft $\left\{\begin{array}{l}\text{at tens.} = 5.9 \text{ at.}\\ \text{" temp. } 297^\circ \text{ Ab. C.}\end{array}\right.$

$\therefore$ from eq. (2)

$$\frac{p_n - p_m}{\gamma'} = \frac{4f}{2g} \cdot \frac{l}{d} \cdot \left[\frac{Q'}{\frac{1}{4}\pi d^2}\right]^2 \text{ whence } d^5 = \frac{4f}{(\frac{1}{4}\pi)^2} \cdot \frac{\gamma' l}{(p_n - p_m)} \cdot \frac{Q'^2}{2g} \text{ .... (5)}$$

$\therefore$ numerically

$$d = \sqrt[5]{\frac{4 \times .00425 \times 0.431 \times 30000 \times (14.74)^2}{(.7854)^2 \left[14.7 \times 144(6.00 - 5.80)\right] 2 \times 32.2}} = 1.23 \text{ feet.}$$

510. (Case II of § 508). Case II. CONSIDERABLE DIFFERENCE OF PRESSURE AT EXTREMITIES OF THE PIPE. FLOW STEADY. Fig. 609. If the difference between end-tensions is comparatively great, we can no longer deal with the whole of the air in the pipe at once, as regards ascribing

To it a mean velocity and mean tension, but must consider the separate laminae, such as AB, to which we may apply eq. (2) of § 609.

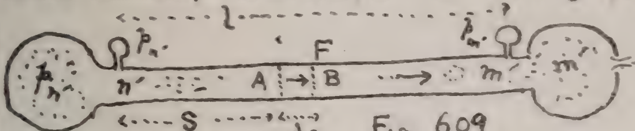


Fig. 609

ISOTHERM. FLOW

Since the $p_{n'} - p_{m'}$, l, γ', and v', of § 609 correspond to the $-dp$, ds, γ, and dv of the present case (short section or lamina) we may write

$$-\frac{dp}{\gamma} = 4f\,\frac{v^2}{d\,2g}\,ds \quad \text{.........} (1)$$

But if G = weight of flow per second, (time unit) we have at any section $F v \gamma = G$ (eq. of continuity) i.e. $v = G \div F\gamma$, whence by substitution in eq. (1) we have

$$-\frac{dp}{\gamma} = \frac{4f}{2gd}\cdot\frac{G^2\,ds}{F^2\gamma^2}\ ; \quad \text{i.e.} -\gamma\,dp = \frac{4fG^2}{2gF^2d}\,ds \quad \text{.....} (2)$$

containing three variables γ, p, and s (= distance of lamina from n'.) As to the dependence of the heaviness γ on the tension p in different laminae, experiment shows that in most cases a uniform temperature is found to exist all along the pipe if properly buried or shaded from the sun; the loss of heat by adiabatic expansion being in great part made up by the heat generated by the friction against the walls of the pipe. This is due to the small loss of tension per unit of length of pipe as compared with occurring in a short discharge pipe or nozzle. Hence we may treat the flow as ISOTHERMAL, and write $p \div \gamma = p_{n'} \div \gamma_{n'}$ (§ 440 Mariotte's Law)

$\therefore \gamma = \dfrac{\gamma_{n'}}{p_{n'}}\,p$ which in eq. 2 enables us to write

$$-p\,dp = \left[\frac{4fG^2}{2gF^2d}\,\frac{p_{n'}}{\gamma_{n'}}\right]ds \ \text{....} \ \therefore -\int_{n'}^{m'} p\,dp = \left[\frac{4fG^2p_{n'}}{2gF^2d\gamma_{n'}}\right]\int_0^l ds \ \text{...} (3)$$

Performing the integration, noting that at n' $p = p_{n'}$, $s = 0$, and at m' $p = p_{m'}$ and $s = l$ we have

$$\frac{1}{2}\left[p_{n'}^2 - p_{m'}^2\right] = \frac{4fl}{2gd}\cdot\frac{G^2}{F^2}\cdot\frac{p_{n'}}{\gamma_{n'}} \ \text{....} \left\{\begin{matrix}\text{ISOTHERMAL}\\ \text{FLOW IN}\\ \text{LONG PIPES}\end{matrix}\right\} \text{....} (4)$$

It is here assumed that the tension at the entrance of the pipe is practically equal to that in the head reservoir; and that at the end (m') to that of the receiving reservoir, which is not strictly true

especially when the corners are not rounded. It will be remembered also that in establishing eq.(2) of § 509 (the basis of the present §) the "inertia" of the gas was neglected, i.e. the change of velocity in passing along the pipe. Hence eq.(4) should not be applied to cases where the pipe is so short, or the difference of end-tensions so great, as to create a considerable difference of velocity at the two ends of the pipe.

Example. A well or reservoir supplies natural gas at a tension of $p_{n'} = 30$ lbs. per sq. inch. Its heaviness at 0° Cent. and one atmos tension is .0350 lbs. per cub. foot. In piping this gas along a level to a town two miles distant, a single four-inch pipe is to be employed, and the tension in the receiving reservoir (by proper regulation of the gas distributed from it) is to be kept equal to 16 lbs. per sq. in (which would sustain a column of water about 2 ft. in height in an OPEN water manometer, Fig. 463). The mean temperature in the pipe being 17° Cent., required the amount (weight) of gas delivered per second. Solve (4) for G and we have

$$G = \frac{1}{4}\pi d^2 \sqrt{\frac{gd}{4fl} \cdot \frac{\gamma_{n'}}{p_{n'}}\left(p_{n'}^2 - p_{m'}^2\right)} \quad \ldots\ldots (5)$$

First, from § 437, with $T_{n'} = T_{m'} = 290°$ Abs. Cent., we compute

$$\frac{p_{n'}}{\gamma_{n'}} = \frac{p_0}{\gamma_0} \cdot \frac{T_{n'}}{T_0} = \frac{14.7 \times 144}{.0350} \cdot \frac{290}{273} = 64240.\ \text{feet.}$$

$\therefore$ with $f = .005$,

$$G = \frac{1}{4}\pi\left(\frac{4}{12}\right)^2 \sqrt{\frac{32.2 \times \frac{4}{12}\left[(30\times 144)^2 - (16\times 144)^2\right]}{4 \times .005 \times 10560 \times 64240}}$$

$\therefore$

$G = 0.2870$ lbs. per second (For compressed atmos. air, under like conditions we should have $G = 0.430$ lbs. per second

Of course the proper choice of the co-efficient f, has an important influence on the result.

Prof. Robinson recommends the value $f = .005$.

CHAP. VII.

Impulse and Resistance of Fluids.

511. THE SO CALLED "REACTION" OF A JET OF WATER FLOWING FROM A VESSEL. In Fig. 610, if a frictionless but water-tight plug B be inserted in the orifice in the vertical side of a vessel mounted on wheels, the resultant action of the water on the rigid vessel (as a whole) consists of its weight G and a force $P' = Fh\gamma$ (in which F = the area of orifice) which is the excess of the horizontal hydrostatic pressures on the vessel wall toward the right (∥ to paper) over those toward the left, since the pressure $P = Fh\gamma$ exerted on the plug is felt by the post C, and not by the vessel. Hence the post D receives a pressure

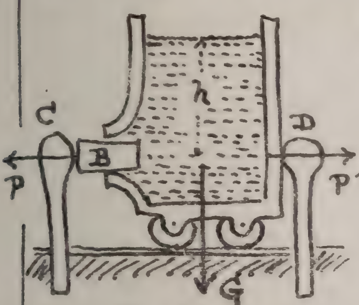


Fig. 610

$$P' = Fh\gamma \quad \text{------} \; (1)$$

Now if D were suddenly removed, assuming no friction at the wheels, the vessel would begin a uniformly accelerated motion toward the right, the accelerating force being $Fh\gamma$, and continue it until the orifice had left the plug. When the plug is out, however, and a steady flow set up thro' the orifice, not only is the pressure $Fh\gamma$ lacking on the left, on account of the orifice, but the sum of the horizontal components, ∥ to paper, of the pressures of the liquid filaments against the vessel wall around the orifice is less than its value before the flow began by an amount $= Fh\gamma$ (for the well-rounded mouth-piece in figure) (see next §). Hence during efflux, the resultant horizontal action of the water on the vessel is a force $2\,Fh\gamma$; i.e., Fig. 611,

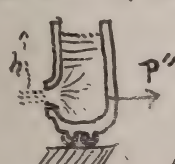


Fig. 611

$$\left.\begin{matrix}\text{THE "REACTION"}\\ \text{OF THE JET}\end{matrix}\right\} = P'' = 2Fh\gamma \quad \text{.....} \; (2)$$

(γ = heaviness of water or other liquid)

§.512. "REACTION" OF A LIQUID JET ON THE VESSEL FROM WHICH IT ISSUES. Instead of showing that the pressures on the vessel close to the orifice are less

than before efflux by a definite amount, it is much simpler to treat the "reaction", or pressure between the base of the jet and the comparatively still water behind the orifice, in a direct manner as follows: Let v = velocity of the jet just in the plane of the orifice Fig. 611; then $v = \sqrt{2gh}$. Also let ΔM represent the mass of the quantity of water which in the short time Δt has had its veloc. changed from zero (just inside the orifice) to v; then the mean acceleration of its motion has been $p'' = \dfrac{v-0}{\Delta t}$ and the corresponding mean force which must have acted between this mass and the vessel is

P'' = mass × acc. $= \dfrac{\Delta M\, v}{\Delta t}$. If Q = vol. of water discharged per unit of time, then $\Delta M = \dfrac{Q\gamma\,\Delta t}{g}$

and since $v = \sqrt{2gh}$ and $Q = Fv = F\sqrt{2gh}$

we have finally } "REACTION" $= P'' = 2Fh\gamma$ .... (3.)

(If the orifice is in "thin plate" we understand F as the area of the contracted section) Practically, we have $v = \phi\sqrt{2gh}$ and hence (3) reduces to

(see § 454) $P'' = 2\phi^2 Fh\gamma$ .... (4)

Example. Weisbach mentions the experiments of Mr. Peter Ewart of Manchester England, as giving the result $P'' = 1.73\,Fh\gamma$ with a well-rounded orifice as in Fig. 611. He also found $\phi = .94$ for the same orifice, so that, eq. (4), theoretically we have $P'' = 2(.94)^2 Fh\gamma = 1.77\ Fh\gamma$

With an orifice in thin plate Mr. Ewart found $P'' = 1.14\,Fh\gamma$. As for a result from eq. 4, we must put, for F, the area of the contracted section $.64\,F$ (§ 454), which with $\phi = .96$ gives

$$P'' = 2\,(.96)^2 .64\,Fh\gamma = 1.18\ Fh\gamma \quad \cdots (5)$$

Both results agree well with experiment.

513. IMPULSE OF A JET OF WATER ON A FIXED CURVED VANE (WITH BORDERS). The jet passes tangentially upon the vane. Fig. 614. A is the stationary nozzle

hich a jet of water of cross section F (area) and velocity = c, impinges tangentially upon the vane, which has plane borders, ∥ to paper, to prevent the lateral escape of the jet. The curve of the vane is not circular necessarily. The vane being smooth, the velocity of the water in its curved path remains = c at all points along the curve.

Fig. 614.

Conceive the curve divided into a great number of small lengths each = ds, and subtending some angle = $d\phi$ from its own centre of curvature, its radius of curvature being = r (different for different ds's) which makes some angle = ϕ with the axis Y ⊥ to original straight jet BA. At any instant of time there is an arc of water AD in contact with the vane, exerting pressure upon it. The pressure dP of any ds of the vane against the small mass of water $Fds\gamma \div g$, then in contact with it is the "deviating" or "centripetal force accountable for its motion in a curve of radius = r, and hence must have a value

$$dP = [F ds \gamma \div g] c^2 \div r \quad \text{......}(1) \quad (\text{see § 76})$$

The opposite and equal of this force is the dP shown in Fig. 614 and is the *impulse* or pressure of this small mass against the vane. Its X component is $dX = dP \sin\phi$. By making ϕ vary from 0 to α, and adding up the corresponding values of dX, we obtain the sum of the X components of the small pressures exerted simultaneously against the vane by the arc of water then in contact with it. I.e., noting that $ds = r d\phi$

$$\int_{\phi=0}^{\phi=\alpha} dX = \int_0^\alpha dP \sin\phi = \frac{F\gamma c^2}{g}\int_0^\alpha \frac{ds \sin\phi}{r} = \frac{F\gamma c^2}{g}\int_0^\alpha \sin\phi \, d\phi = \frac{F\gamma c^2}{g}\Big[-\cos\phi\Big]_0^\alpha$$

∴ the X-IMPULSE AG.NST FIXED VANE $\Big\} = \dfrac{F\gamma c^2}{g}[1-\cos\alpha] = \dfrac{Q\gamma c}{g}[1-\cos\alpha] \text{......}(1)$

in which $Q = Fc$ = vol. of water which passes thro' the nozzle (and also = that passing over the vane, in this case)

per unit of time, and α = angle between the direction of the stream leaving the vane (i.e. at D) and its original direction BA of the jet; i.e. α = total angle of deviation. Similarly the sum of the Y components of the dP's of Fig. 614 may be shown to be

$$\left.\begin{array}{l}\text{Y-IMPULSE ON}\\ \text{FIXED VANE:}\end{array}\right\} = \int_0^\alpha dP\cos\phi = \frac{Q\gamma c}{g}\sin\alpha \quad \text{.......}(2)$$

Hence the resultant impulse on the vane is a force

$$P'' = \sqrt{X^2+Y^2} = \frac{Q\gamma c}{g}\sqrt{2(1-\cos\alpha)} \quad \text{.....}(3)$$

and makes an angle α', Fig. 615; with the direction BA, that

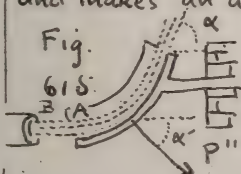


Fig. 615.

$$\tan\alpha' = \frac{Y}{X} = \frac{\sin\alpha}{1-\cos\alpha} \quad \text{........}(4)$$

For example if $\alpha = 90°$, then $\alpha' = 45°$; while if $\alpha = 180°$, Fig. 616, we have $\alpha' = 0°$, i.e. P'' is ∥ to the jet BA, and its value is

$$P'' = 2Q\gamma\frac{c}{g}$$

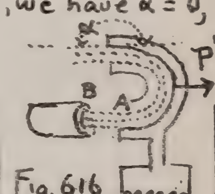


Fig. 616.

514. IMPULSE OF A JET ON A FIXED SOLID OF REVOLUTION WHOSE AXIS IS PARALLEL TO JET.

If the curved vane, with borders, of the preceding § be replaced by a solid of revolution, Fig. 617, with its axis in line of the jet, the resultant pressure of the jet upon it will simply be the sum of the X-components (i.e. ∥ to BA) of the pressures on all elements of the surface at a given instant i.e.

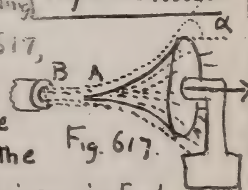


Fig. 617.

$$X = P'' = Q\gamma\frac{c}{g}(1-\cos.\alpha) \quad \text{.......}(5)$$

while the components ⊥ to X, all radiating from the axis of the solid, neutralize each other. For a FIXED PLATE, then, Fig. 618, at right angles to the jet we have the impulse (with $\alpha = 90°$)

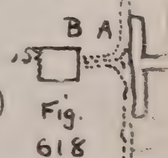


Fig. 618

$$P'' = \frac{Q\gamma}{g}c = \frac{Fc^2}{g}\gamma = 2\cdot F\frac{c^2}{2g}\gamma \quad \text{...}(6)$$

The experiments of Bidone (made in 1838) confirm eq.(6) quite closely. Eq.(6) is applicable to the theory of Pitot's Tube (see § 491), Fig. 619, if we consider the edge of the tube plane and quite moist. The water in the tube is at rest, and its section at A may be treated as a flat vertical plate receiving not only the hydrostatic pressure $Fx\gamma$, due to the depth x below the surface, but a continuous impulse $P'' = Fc^2\gamma \div g$, (see eq. 6). For the equilibrium of the end A, of the stationary column AD, we must have, ∴,

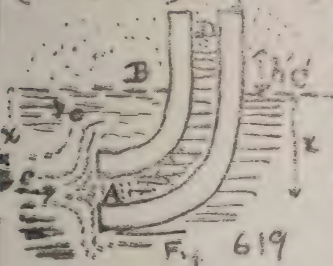


Fig. 619

$$Fx\gamma + \frac{Fc^2}{g}\gamma = Fx\gamma + Fh'\gamma \text{ ....i.e. } c = \sqrt{gh} \text{ ......(7)}$$

(See § 491 for experimental support of this relation); F = sectional area at A.

If the solid of revolution is made cup-shaped, as in Fig. 620 we have (as in Fig 616 also) $\alpha = 180°$ and ∴ from eq. (6'),

$$(8) \text{.....} P'' = 2Q\gamma\frac{c}{g} = \frac{2Fc^2}{g}\gamma = \frac{4Fc^2}{2g}\gamma$$

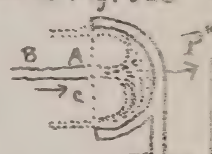


Fig. 620

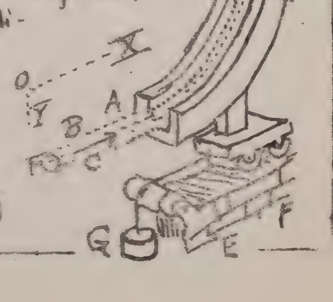
Example Fig. 620. If $c = 30$ ft. per sec. and the jet has a diameter of 1 inch, the liquid being water, so that $\gamma = 62.5$ lbs. per cub. ft.; we have [Ft. lb. sec.]

$$\text{the impulse (force)} = P'' = \frac{2\,\frac{\pi}{4}\left(\frac{1}{12}\right)^2 900 \times 62.5}{32.2} = 19.05 \text{ lbs.}$$

Experiment would probably show a smaller result; somewhat.

515. IMPULSE OF A LIQUID JET UPON A MOVING VANE HAVING LATERAL BORDERS AND MOVING IN THE DIRECTION OF THE JET. Fig. 621. The vane has a motion of translation (§ 108) in the same direction as the jet. Call this the axis X. It is moving with a velocity v away from the jet (or, if toward the jet, v is negative). We consider v constant its acceleration being prevented by a proper resistance (such as a weight = G)

Fig. 621



to balance the X-components of the n.c-pressures. Before coming in contact with the vane, which it does *tangentially* (to avoid sudden deviation) the absolute velocity (§ 83) of the water in the jet $= c$, while its velocity relatively to the vane AB is $= c - v$, this relative velocity remains the same along the vane; for friction is disregarded and the absol. veloc. of each point of the vane is of the same amount and direction as of any other. [N.B. If the motion of the vane were rotary about an axis ⊥ to AB (or to c), this relative velocity would be different at different points, See Hydraulic Motors. If the radius of motion of the point A, however, were quite large compared with with the projection of AD upon this radius, the relative veloc. would be approx $= c - v$ at all parts of vane.] It is evident that the analysis of § 513 is applicable here, putting $c - v$ for the c of that article; whence (from eq. 1 § 513)

COMPONENT OF IMPULSE IN DIRECTION OF JET $\Big\} = P_x = \dfrac{F\gamma}{g}(c-v)^2\left[1 - \cos\alpha\right] \ldots (1)$

(where α is the angle of total deviation, of stream leaving the vane, from its original direction) and is seen to be proportional to the square of the relative velocity. F is the sectional area of jet, and γ the heaviness (§ 7) of the liquid. The Y component (or P_Y) of the resultant impulse is counteracted by the support EF, Fig. 621. *Hence, for a uniform motion to be maintained*, with a given velocity $= v$, the weight G must be made $= P_x$ in eq. (1). (We here neglect friction and suppose the jet to preserve a practically horizontal direction for an indefinite distance before meeting the vane. If this uniform motion is to be toward the jet, v will be negative in eq. (1), making P_x (and ∴ G) larger than for a positive v of same numerical value.

As to the *doing of work* [§§ 128 etc.], or exchange of energy, between the two bodies, jet and vane, during a uniform motion away from the jet P_x exerts a power of

$$(\text{POWER} =) \ldots L = P_x v = \frac{F\gamma}{g}(c-v)^2 v\left[1 - \cos\alpha\right] \ldots (2)$$

in which L denotes the number of units of work done per unit of time by P_x, i.e. the power, (§ 130) exerted by P_x.

If v is negative, call it $-v'$, and we have the

$$\left.\begin{array}{l}\text{POWER EXPENDED}\\ \text{BY VANE UPON JET}\end{array}\right\} = P_x v' = \frac{F\gamma}{g}(c+v')^2 v'[1-\cos\alpha] \quad \ldots (3)$$

Of course, practically, we are more concerned with eq. (2) than with (3). The power L in (2) is a maximum for $v = \frac{1}{3}c$; but in practice, since a single moving vane or float cannot utilize the water of the jet as fast as it flows from the nozzle, let us conceive of a succession of vanes coming into position consecutively in front of the jet, all having the same velocity v; then the portion of jet intercepted between two vanes is at liberty to finish its work on the front vane, while additional work is being done on the hinder one; i.e., the water will be utilized as fast as it issues from the nozzle.

With such a series of vanes, then, we may put $Q = Fc$, the volume of flow per unit of time from the nozzle, in place of $F(c-v)$ = the vol. of flow " " " " over the vane, in eq. (2), whence,

$$\left.\begin{array}{l}\text{POWER EXERT-}\\ \text{ED ON SERIES OF VANES}\end{array}\right\} = L' = \frac{Q\gamma}{g}[1-\cos\alpha](c-v)v \quad \ldots (4)$$

Making v variable, and putting $dL' \div dv = 0$, whence $c - 2v = 0$, we find that for $v = \frac{1}{2}c$, L', the power, is a maximum. Assuming different values for α, we find that for $\alpha = 180°$, i.e. by the use of a semicircular vane, or of a hemispherical cup, Fig. 622, with a point in middle, $1-\cos\alpha$ is a max. $= 2$; whence, with $v = \frac{1}{2}c$, we have, as the maximum power

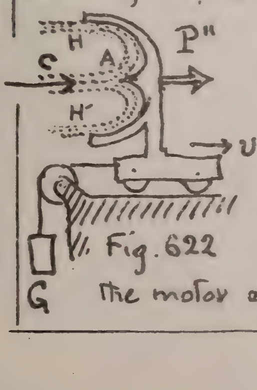


Fig. 622

$$L'_{max.} = \frac{Q\gamma}{g}\cdot\frac{c^2}{2} = \frac{M'c^2}{2} \quad \ldots \left\{\begin{array}{l}\alpha = 180°\\ v = \frac{1}{2}c\end{array}\right\} \quad \ldots (5)$$

in which M' denotes the mass of the flow per unit of time from the stationary nozzle. Now $\frac{M'c^2}{2}$ is the entire kinetic energy furnished per unit of time by the jet; hence the motor of Fig. 622 (series of cups) has a theoretical

efficiency of unity, utilizing all the energy of the water. If this is true, the absolute velocity of the particles of liquid where they leave the cup, or vane, should be <u>zero</u>, which is seen to be true, as follows: At H, or H', the velocity of the particles relatively to the vane is $= c - v =$ what it was at A, and $\therefore$ is $= c - \frac{c}{2} = \frac{c}{2}$; hence at H the absolute velocity is $w =$ (rel. veloc. $\frac{c}{2}$ toward left) $-$ (vel. $\frac{c}{2}$ of vane toward right) $= 0$; Q. E. D. For $v >$ or $< \frac{1}{2}c$ this max. efficiency will not be attained.

516. THE CALIFORNIA "HURDY-GURDY," OR PELTON WHEEL. The efficiency of unity in the series of cups just mentioned is in practice reduced to 80 or 85 per cent. from friction and lateral escape of water. The Pelton wheel or California "Hurdy-gurdy," shown, in principle only, in Fig. 623, is designed to utilize the mechanical principle just presented and yields results confirming the above theory, viz., that with the linear velocity of the cup-centres regulated to equal $\frac{c}{2}$, and with $\alpha = 180°$, the efficiency approaches unity or 100 per cent.

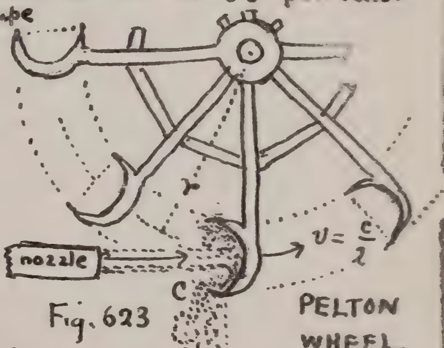


Fig. 623

PELTON WHEEL. (Ideal.)

This wheel was invented to continue to utilize small jets of very great velocities (c) in regions just deserted by "hydraulic mining" operators. Although c is great, still, by giving a large value to r, the radius of the wheel (to cup centres) the making $v = \frac{c}{2}$ does not necessitate an inconveniently great speed of rotation (i.e. revols. per unit of time). The plane of the wheel may be in any convenient position.

Example. If the jet in Fig. 623 has a velocity $c = 60$ ft. per second and is delivered thro' a 2 inch nozzle, the

total power due to the kinetic energy of the water is (ft lb. sec)

$$\frac{Q\gamma}{g}\cdot\frac{c^2}{2} = \frac{1}{32.2}\cdot\frac{\pi}{4}\left(\frac{2}{12}\right)^2 \times 60 \times 62.5 \times \frac{1}{2} \times 3600 = 4566.9 \left\{\begin{matrix}\text{ft. lbs}\\ \text{per sec}\end{matrix}\right.$$

and if, by making the velocity of the cups $= \frac{c}{2} =$ 30 ft. per sec., 85 per cent. of this power can be utilized, the power of the wheel at this most advantageous velocity is

$$L = .85 \times 4566.9 = 3881.\text{ ft lbs. per sec.} = 7.05 \text{ Horse power}$$

[Since $3881. \div 550 = 7.05$] (§132). For a cup-velocity of 30 ft. per sec., if we make the radius, r, = to 10 feet, the angular velocity of the wheel will be $\omega = v \div r = 3.0$ radians (for radian see Ex. in § 410; for ang. vel., §110), which nearly $= \pi$, thus implying nearly a half-revol. per-sec.

517. OBLIQUE IMPACT OF A JET ON A MOVING PLATE HAVING NO BORDER.

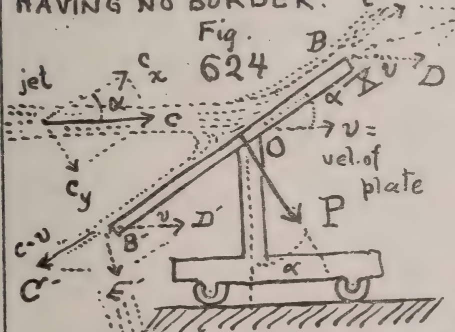


Fig. 624

The plate has a motion of translation with a uniform veloc. $= v$ in a direction ∥ to jet, whose velocity is $= c$. At O the filaments of liquid are deviated, so that in leaving the plate their particles are all found in the moving plane BB′ of the plate surface, but the respective absolute velocities of these particles depend on the location of the point of the plate where they leave it, being found by forming a diagonal on the relative veloc. $c-v$ and the velocity v of the plate. For example at B the absolute velocity of a liquid particle is $w = BE = \sqrt{v^2 + (c-v)^2 + 2c(c-v)\cos\alpha}$ while at B′ it is $B'E' = w' = \sqrt{v^2 + (c-v)^2 - 2c(c-v)\cos\alpha}$; but evidently the component ⊥ to plate (the other compon. being ∥) of the absolute velocities of all particles leaving the plate, is the same and $= v\sin\alpha$. The skin-friction of the liquid on the plate being neglected, the resultant impulse of the jet against the plate must be NORMAL to its surface.

and its amount, P, is most readily found as follows:

Denot by ΔM the mass of the liquid passing over the plate in a short time Δt, resolve the absolute velocities of all the liquid particles, before and after deviation, into components ⊥ to the plate (call this direction Y) and ∥ to the plate. Before meeting the plate the particles composing ΔM have a velocity in the direction of Y of $c_y = c \sin\alpha$; on leaving the plate a vel. in dir. of Y of $v \sin\alpha$; they have ∴ lost an amount of Y veloc. $= (c-v)\sin\alpha$ in time Δt; i.e. they have suffered an average retardation (or neg. acc.) in a Y-direction of

$$p_Y = \left\{\begin{matrix}\text{neg. accel-}\\ \text{eration} \parallel \text{to Y}\end{matrix}\right\} = \frac{(c-v)\sin\alpha}{\Delta t}.$$

Hence the resistance in direction of Y (i.e. the equal and opp. of P in figure) must be $P_Y = \text{mass} \times \text{Y-accel} = \frac{\Delta M}{\Delta t}(c-v)\sin\alpha$; and ∴, since $\frac{\Delta M}{\Delta t} = M = \frac{Q\gamma}{g}$ = mass of liquid passing over the plate per unit of time (not that issuing from nozzle) we have

$$\left.\begin{matrix}\text{IMPULSE OR}\\ \text{PRESSURE ON PLATE}\end{matrix}\right\} = P = \frac{Q\gamma}{g}(c-v)\sin\alpha = \frac{F\gamma}{g}(c-v)^2\sin\alpha \quad \ldots (1)$$

in which F = sectional area of jet before meeting plate.

[N.B. Since eq. (1) can also be written $P = Mc\sin\alpha - Mv\sin\alpha$ and $Mc\sin\alpha$ may be called the *Y-Momentum before contact*, while $Mv\sin\alpha$ is the *Y-Momentum after contact*, (of the mass passing over plate per unit of time) this method is sometimes said to be founded on the principle of momentum which is nothing more than the relation that the accel. force in any direction = mass × acceleration in that direction; e.g. $P_x = Mp_x$; $P_y = Mp_y$; see § 74]

If we resolve P, Fig. 624, into components, one, P′, ∥ to

to the direction of motion (v and c), and the other, P'', ⊥ to the same, we have

$$P' = P\sin\alpha = \frac{Q\gamma}{g}(c-v)\sin^2\alpha \quad \cdots\cdots (2)$$

$$\text{and} \quad P'' = P\cos\alpha = \frac{Q\gamma}{g}(c-v)\sin\alpha\cos\alpha \quad \cdots\cdots (3)$$

($Q = F(c-v)$ = vol. passing over the plate per unit of time) The force P'' does no work, while the former, P', does an amount of work $P'v$ per unit of time, i.e. exerts a

$$\underset{\text{(ONE PLATE)}}{\text{POWER}} = L = P'v = \frac{Q\gamma}{g}(c-v)v\sin^2\alpha \quad \cdots\cdots (4)$$

If instead of a single plate, a series of plates, forming a regular succession, is employed, then as in a previous paragraph we may replace Q, $= F(c-v)$, by $Q' = Fc$, obtaining as the

$$\left.\begin{matrix}\text{POWER EXERTED BY JET}\\ \text{ON SERIES OF PLATES}\end{matrix}\right\} = L' = \frac{Fc\gamma}{g}(c-v)v\sin^2\alpha \quad \cdots\cdots (5)$$

For $v = \frac{c}{2}$ and $\alpha = 90°$ we have $L'_{max} = \frac{1}{2}\frac{Fc\gamma}{g}\frac{c^2}{2} = \frac{1}{2}\frac{M'c^2}{2} \quad ..(6)$

= only half the kinetic energy (per time-unit) of the jet.

518. RIGID PLATES MOVING IN A FLUID, TOTALLY SUBMERGED. FLUID MOVING AGAINST A FIXED PLATE. IMPULSE AND RESISTANCE. If a thin flat rigid plate have a motion of uniform translation with velocity $= v$ through a fluid which completely surrounds it, Fig. 625, a resistance is encountered (which must be overcome by an equal an opposite forward force not shown in figure to preserve the uniform motion) consisting of a normal component N, ⊥ to plate, and a (small) tangential component, or skin-friction, T, ∥ to plate.

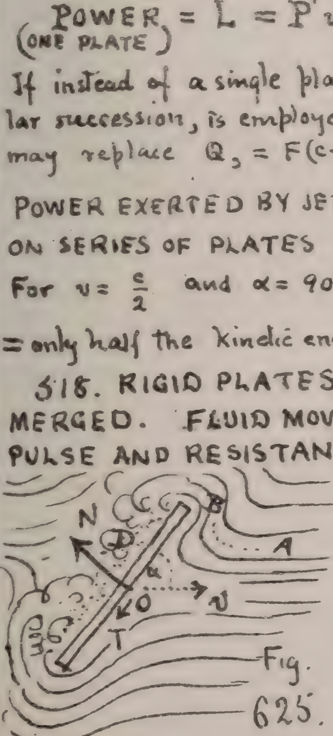


Fig. 625.

Unless the angle α, between the surface of plate and the direction of motion $O \ldots v$, is very small, i.e., unless the plate is moving nearly edgewise thro' the fluid, N is usually much $> T$. The skin-resistance between a solid and fluid has already been spoken of in § 469

When the plate and fluid are at rest the pressures

on both sides are normal and balance each other being ordinary static fluid pressures. When motion is in progress, however, the normal pressures on the front surface are increased by the components, normal to plate, of the centrifugal forces of the curved filaments (such as AB) or "stream lines", while on the back surface, D, the fluid does not close in fast enough to produce a pressure equal to that (even) of rest. In fact if the motion is sufficiently rapid and the fluid is inelastic (a liquid) a vacuum may be maintained behind the plate, in which case there is evidently no forward pressure from behind.

Whatever pressure exists on the back acts, of course, to diminish the resultant resistance. The water on turning the sharp corners of the plate is broken up into eddies forming a "wake" behind. From the accompaniment of these eddies, the resistance in this case (at least the component N normal to plate) is said to be due to eddy-making; though logically we should say, rather, that the body does not derive the assistance (or negative resistance) from behind which it would obtain if eddies were not formed, i.e., if the fluid could close in behind in smooth curved stream-lines symmetrical with those in front.

The *heat* corresponding to the change of temperature produced in the portion of fluid acted on, by the skin friction and by the mutual friction of the particles in the eddies, is the equivalent of the work done (or energy spent) by the motive force in maintaining the uniform motion (§ 149)

If the fluid is sea-water the results of Col. Beaufoy's experiments are applicable; viz.:

The resistance, per square foot of area, sustained by a submerged plate moving normally to itself [i.e. $\alpha = 90°$] in sea-water with a velocity of $v = 10$ ft. per second is 112. lbs. He also asserts that for other velocities the resistance varies as the square of the velocity. This latter fact we would be led to suspect from the results obtained in § 517 for the impulse of

jets; also in § 514 (see eq. 6). Also that when the plate moved obliquely to its normal (as in Fig. 625) the resistance was nearly equal to (the resistance, at same veloc., when $\alpha = 90°$) $\times$ the sine of the angle α; also that the depth of submersion had no influence on the resistance.

Confining our attention to a plate moving normally to itself, Fig. 626,

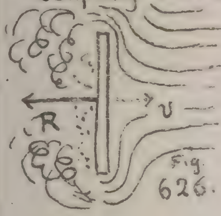


Fig. 626.

let F = area of plate, γ = heaviness (§ 7) of the fluid, v = the uniform velocity of plate, and g = the acceleration of gravity (= 32.2 for the foot and second; = 9.81 for the metre and second) Then from the analogy of eq. (b) § 514, where the veloc. c of the jet against a stationary plate corresponds to the veloc. v of the plate in the present case moving through a fluid at rest, we may write

$$\left.\begin{array}{l}\text{RESISTANCE OF FLUID}\\ \text{TO MOVING PLATE}\end{array}\right\} = R = \zeta F \gamma \frac{v^2}{2g} \quad \cdots\cdots \left\{\begin{array}{l}v \text{ normal}\\ \text{to plate}\end{array}\right. \quad (1)$$

and similarly for the *impulse* of an *indefinite stream* of fluid against a fixed plate ($\perp$ to veloc. of stream), v being the velocity of the current,

$$\left.\begin{array}{l}\text{IMPULSE OF FLUID}\\ \text{UPON FIXED PLATE}\end{array}\right\} = P = \zeta' F \gamma \frac{v^2}{2g} \quad \cdots\cdots \left\{\begin{array}{l}v \text{ normal}\\ \text{to plate}\end{array}\right\} \cdots (2)$$

The $2g$ is introduced simply for convenience; since, having v given, we may easily find $v^2 \div 2g$ from a table of velocity-heads; and also (a ground of greater importance) since the co-efficients ζ and ζ' which depend on experiment are evidently abstract numbers in the present form of these equations; (for R and P are forces, and $F\gamma v^2 \div 2g$ is the weight (force) of an ideal prism of fluid; hence ζ and ζ' must be abstract numbers)

From Col. Beaufoy's experiments (see above) we have for sea-water [ft. lb. sec.] putting $R = 112$ lbs., $F = 1$ sq. ft., $\gamma = 64$ lbs. per cub. ft, and $v = 10$ ft. per second,

$$\zeta = \frac{2 \times 32.2 \times 112}{1.0 \times 64 \times 10^2} = 1.13$$

{Hence in eq. (1) for sea-water we may put $\zeta = 1.13$ (with $\gamma = 64$. lbs. per cub. ft.}

From the experiments of Du Buat and Thibault, Weisbach computes that for the plate of Fig. 626, moving through either water or air, $\zeta = 1.25$ for eq. (1), in which

the γ for air must be computed from § 437; while for the impulse of water or air on fixed plates he obtains $\zeta' = 1.86$ for use in eq. (2) (The results of experiment in this section seem uncertain and conflicting.) For great velocities ζ and ζ' are greater for air than for water, since air, being compressible, is of greater heaviness in front of the plate than the γ which would be computed for the given temperature and barometric height for use in eqs. (1) and (2). [For Example see § 520]

519. WIND-PRESSURE on the surface of a roof inclined at an angle $= \alpha$ with the horizontal i.e. with the direction of the wind is usually estimated according to the empirical formula:

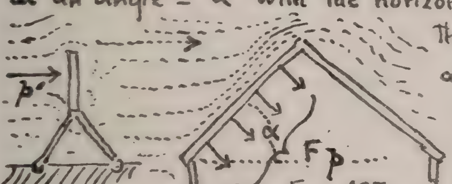


Fig. 627.

$$p = p'(\sin\alpha)^{[1.84\cos\alpha - 1]} \quad \dots\dots\dots \quad (1)$$

in which p = pressure of wind against a vertical surface (⊥ to wind) per unit of area, and p' = that against the inclined plane at the same velocity (per unit of area). For a value of $p' = 40$ lbs. per square foot (as a maximum) we have the following values for p computed from (1)

For $\alpha =$	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°
$p =$ lbs. sq. ft	5.2	9.6	14	18.3	22.5	26.5	30.1	33.4	36.1	38.1	39.6	40.

520. NUMERICAL EXAMPLE UNDER § 518. When a blade (of a paddle-wheel) having an area of 6 sq. ft., is in its lowest position, its velocity (relatively to water, not to vessel) being then 5 ft. per sec., what resistance is it overcoming, in salt water?

From eq. (1) of § 518 with $\zeta = 1.13$ and $\gamma = 64$ lbs. per cubic foot, we have

$$\text{(ft. lb. sec.)} \quad R = \frac{1.13 \times 6 \times 64 \times 25}{2 \times 32.2} = 169.4 \text{ lbs.}$$

If on the average there may be considered to be three paddles always overcoming this resistance on each side of the boat, then the work lost (work of "slip") in overcoming these

resistances per second (i.e. power lost) is

$I_1 = [6 \times 169.4]$ lbs $\times 5$ ft. per sec. $= 5082$ ft. lbs. per sec.

or 9.24 horse power (since $5082 \div 550 = 9.24$)

If, further, the velocity of the boat is uniform and = 20 ft. per sec., the resistance of the water to the progress of the boat at this speed being 6×169.4, i.e., 1016.4 lbs., the power expended in actual propulsion is

$I_2 = 1016.4 \times 20 = 20328$ ft.-lbs. per sec. Hence the power expended in both ways (usefully in propulsion, uselessly in "slip") is $I_2 + I_1 = 25410$ ft.-lbs. per sec. $= 46.2$ H.P.

Of this, 9.24 H.P., or about 20 per cent., is lost in "slip".

521. RESISTANCE OF STILL WATER TO MOVING BODIES, COMPLETELY IMMERSED. This resistance depends on the shape, position, and velocity of the moving body, and also upon the roughness of its surface. If it is pointed at both ends (Fig. 628) with its axis parallel to the velocity, v, of its uniform motion, the stream-lines on closing together smoothly at the hinder extremity, or stern, B, exert normal pressures against the surface of the portion CD..B whose longitudinal components approximately balance the corresponding components of the normal pressures on CD....A; so that the resistance R, which must be overcome to maintain the uniform velocity v, is mainly due to the skin-friction alone, distributed along the external surface of the body; the resultant of these resistances is a force R acting in the line AB of symmetry (supposing the body symmetrical about the direction of motion)

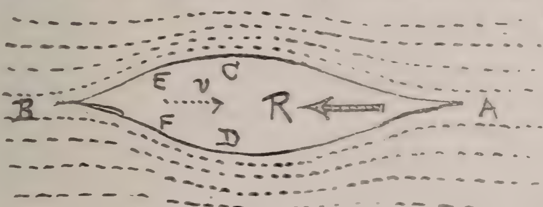


Fig. 628

If, however, Fig. 629, the stern, CD...B, is too bluff,

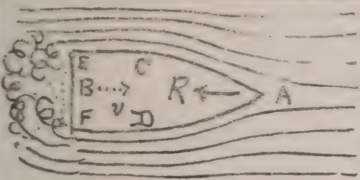
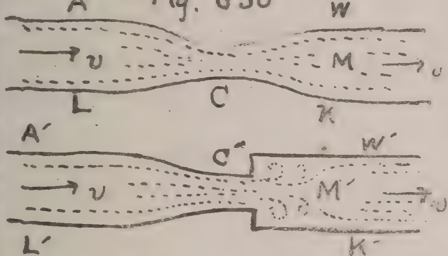


Fig. 629

— eddies are formed round the corners E and F, and the pressure on the surface E...F is much less than in Fig. 628; i.e., the water pressure from behind is less than the backward (longitudinal) pressures from in front, and thus the resultant resistance R is due partly to skin-friction and partly to "eddy-making"

NOTE. The diminished pressure on EF is analogous to the loss of pressure of water (flowing in a pipe) after passing a narrow section the enlargement from which to the original section is sudden. E.g., Fig. 630, supposing the velocity v and pressure p (per unit-area) to be the same respectively at A and A′, in the two pipes shown, with diameter AL = WK = A′L′ = W′K′; then the pressure at M = that at A (disregarding skin-friction) whereas that at M′ is considerably less than that at A′ on account of the head lost in the sudden enlargement. (See also Fig. 562)

Fig. 630



It is ∴ evident that bluffness of stern increases the resistance much more than bluffness of bow.

In any case experiment shows that for a given body, symmetrical about an axis and moving through a fluid (not only water, but any fluid) in the direction of its axis with a uniform velocity = v, we may write the resistance

$$R = (\text{resistance at vel. } v) = \zeta F \gamma \frac{v^2}{2g} \quad \ldots\ldots (1)$$

as in preceding §3. F = area of the greatest section, ⟂ to axis, of the external surface of body (not of the

substance) i.e. the sectional area of the circumscribing cylinder (cylinder in the most general sense) with elements ∥ to axis of body. γ = the heaviness (§ 7) of the fluid, and v = velocity of motion; while ζ is an abstract number dependent on experiment.

According to Weisbach, who cites different experimenters, we can put for SPHERES moving in water ζ = about 0.55
for cannon-balls " in water $\zeta = .467$ { another experim'r.
According to Robins and Hutton, for SPHERES in AIR

For $v =$	1	5	25	100	200	300	400	500..	meters per sec.
$\zeta =$	.59	.63	.67	.71	.77	.88	.99	1.04	

For musket balls in the air, Piobert found

$$\zeta = 0.451\,(1 + 0.0023 \times \text{veloc. in metres per sec.})$$

From Du Buat's experiments, for the resistance of water to a right prism moving endwise and of length $= l$

for $l : \sqrt{F} =$	0	1	2	3
$\zeta =$	1.25	1.26	1.31	1.33

Borda claimed that.

For a circular cylinder moving ⊥'ly to its axis, ζ is one-half as much as for the circumscribing right parallelopiped moving with two faces ∥ to direction of motion.

Example. The resistance of the air at a temperature of freezing and tension of one atmosphere to a musket-ball $\frac{1}{2}$ inch in diameter when moving with a velocity of 328 ft. per sec. is thus determined by Piobert's formula, above;

$$\zeta = 0.451\,(1 + .0023 \times 100) = 0.554 \quad \therefore, \text{ from eq. (1)},$$

§ 394

$$R = .554 \times \frac{\pi}{4}\left(\frac{\frac{1}{2}}{12}\right)^2 \times .0807 \times \frac{328 \times 328}{64.4} = 0.1018 \text{ lbs.}$$

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522. ROBINSON'S CUP ANEMOMETER (for measuring the velocity of the wind) consists of four hemispherical cups of thin metal, Fig. 631, set in a circle, all facing the same way tangent to the circle, and so mounted on a light but rigid frame-work as to be capable of rotating, with as

Fig 631

little friction as possible on a vertical axis. When in a current of air (or other fluid) the whole frame-work soons assumes a uniform motion of rotation (with a corresponding constant linear velocity v'' of the centre of each cup) bearing a definite ratio to the velocity v' of the particles of fluid (i.e. v' = veloc. of wind). Since experiment shows that for the same <u>relative</u> <u>velocity</u> v (of fluid with respect to cup, or vice versâ) the impulse or resistance as the case may be, is about $2\frac{1}{2}$ times as great when the hollow is presented to the current as when the convex side is made to face the stream of air, we may compute roughly the value of this uniform v'' for a given v', neglecting the friction of axle and the influence of the current on those cups the planes of whose openings are ∥ to the current for the instant.

If then we put the impulse or resistance with the hollow in front

$$P_h = \zeta_h F \gamma \frac{v^2}{2g} \quad \text{.............. (1)}$$

and that when the convex surface is in front

$$P_c = \zeta_c F \gamma \frac{v^2}{2g} \quad \text{......... (2)}$$

we may write, approximately, $\zeta_h = 2.5\, \zeta_c$ .... (3)

Regarding only the two cups A and B, which at a definite instant are moving (their centres) in lines ∥ to the direction of the wind, it is evident that the motion of the cups does not become uniform until the relative velocity $v' - v''$ of the wind and cup A (retreating before the wind) is so small, and the relative velocity $v' + v''$, with which B advances to meet the wind, is so great, that the

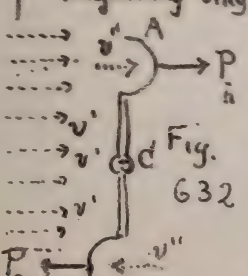

Fig. 632

impulse of the wind on A equals the resistance encountered by B, these forces, P_h and P_c, respectively, having equal lever-arms about the axis. We may ∴ write for uniform rotary motion, since the v of A $= v' - v''$, & v of B $= v' + v''$,

$$\frac{F\gamma}{2g} \zeta_h (v' - v'')^2 = \frac{F\gamma}{g} \zeta_c (v' + v'')^2 \quad \text{.......(4)}$$

i.e., see eq.(3),

$$2.5 (v' - v'')^2 = (v' + v'')^2 ; \quad \text{or } 5\left[\frac{v'}{v''} - 1\right]^2 = 2\left[\frac{v'}{v''} + 1\right]^2 \quad \text{.....(5)}$$

Solving for the ratio $v' : v''$ we find $v' : v'' = 4.4$ or $v' = 4.4 v''$ i.e. for an observed velocity v'' of the cup centres, the velocity of the wind is about four or five times as great.

Experiment shows, however, that the ratio is variable; that for moderate winds it = from 2.5 to 6; for very strong winds as high as 10 or 11

523. RESISTANCE OF SHIPS. We shall first suppose the ship to be towed at a uniform speed, i.e. to be without means of self-propulsion (under water). This being the case it is found that at moderate velocities, (under six miles per hour), the ship being of "fair" form (i.e. the hull tapering both at bow and stern, under water) the resistance in still water is almost wholly due to skin-friction, "eddy-making" being done away with largely by avoiding a bluff stern.

When the velocity is greater than about six miles an hour the resistance is ... much larger than would be accounted for by skin-friction alone, and is found to be connected with the surface-disturbance or waves produced by the motion of the hull in (originally) still water. The recent experiments of Mr. Froude and his son at Torquay England with models in a tank 300 feet long have led to important rules [See Mr. White's Naval Architecture] of so proportioning not only the total length of a ship of given displacement, but the length of the ENTRANCE (forward tapering part of hull) and length of RUN (hinder " " " "), as to secure a minimum "wave-making" resistance, as this source of re-

sistance is called.

To quote from Mr. White [p. 460 of his "Naval Architecture", London, 1882] "Summing up the foregoing remarks, it appears:

"(1) That *frictional resistance*, depending upon the area of the immersed surface of a ship, its degree of roughness, its length, and (about) the square of its speed, is not sensibly affected by the forms and proportions of ships: unless there be some unwonted singularity of form, or want of fairness. For *moderate speeds*, this element of resistance is by far the most important: for *high* speeds it also occupies an important position — from 50 to 60 per cent. of the whole resistance, probably, in a very large number of classes, when the bottoms are *clean*; and a larger percentage when the bottoms become foul."

"(2) That *eddy-making* resistance is usually small, except in special cases, and amounts to 8 or 10 per cent of the frictional resistance. A defective form of stern causes largely increased eddy-making."

"(3) That *wave-making* resistance is the element of the total resistance which is most influenced by the forms and proportions of ships. Its ratio to the frictional resistance, as well as its absolute magnitude, depend on many circumstances; the most important being the forms and lengths of the entrance and run, in relation to the intended full speed of the ship. For every ship there is a limit of speed beyond which each small increase in speed is attended by a disproportionate increase in resistance; and this limit is fixed by the lengths of the entrance and run — the "wave-making features" of a ship"

"The sum of these three elements constitute the total resistance offered by the water to the motion of a ship towed thro' it, or propelled by sails; in a steamship there is an "augment" of resistance due to the action of the propellers."

In the case of a screw propeller at the stern the augment to the resistance varies from 20 to 45 per cent. of the "tow-rope resistance", on account of the presence and action of the propeller itself; since its action relieves the stern of some of the forward hydrostatic pressure of the water closing in around it. Still, if the screw is placed

far back of the stern, this augment is very much diminished; but such a position involves risks of various kinds and is rarely adopted.

We may compute approximately the resistance of the water to a ship propelled by steam at a uniform velocity v, in the following manner: Let L denote the power developed in the engine cylinders; whence, allowing 10 per cent. of L for engine friction, and 15 % for "work of slip" of the propeller-blades, we have remaining $0.75\,L$, as expended in overcoming the resistance R through a distance $= v$ each unit of time. i.e.

(approx.) $$0.75\,L = Rv \quad \text{--- § 132 ---} \quad (1)$$

Example. If 3000 indicated H.P. is exerted by the engines of a steamer at a speed of 15 miles per hour (= 22 ft. per sec.) we have (with above allowance for slip and engine friction)

[Foot-lb.-sec.] $\quad \frac{3}{4} \times 3000 \times 550 = R \times 22 \quad \therefore R = 56250$ lbs.

Further, since R varies (roughly) as the square of the velocity, and can $\therefore$ be written $R = (\text{Const.}) \times v^2$, we have from (1)

$$L = \text{a constant} \times v^3 \quad \ldots\ldots \quad (2)$$

as a roughly approximate relation between the speed and the power necessary to maintain it uniformly; In view of eq. (3), involving the cube of the velocity as it does, we can understand why a large increase of power is necessary to gain a proportionally small increase of speed.

§ 523. "TRANSPORTING POWER" OF A CURRENT. This is sometimes stated to vary as the *sixth* power of the velocity of the current, by which statement is meant, more definitely, the following: Fig. 633. Suppose a row of cubes (or other solids of similar form) of many sizes, all of the same heaviness, and similarly situated, to be placed on the horiz. bottom of a trough and

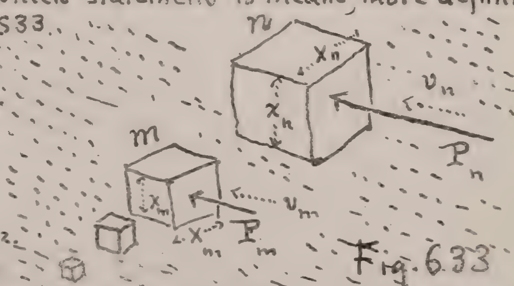

Fig. 633

them exposed to a current of water, completely immersed. Suppose the co-efficient of friction between the cubes and the trough-bottom to be the same for all. Then, as the current is given greater and greater velocity, v, the impulse P_m (corresponding to a particular veloc. v_m) against some one, m, of the cubes will be just sufficient to move it, and at some higher veloc. v_n the impulse, P_n against some larger cube, n, will be just sufficient to move it, in its turn. We are to prove that $P_m : P_n :: v_m^6 : v_n^6$.

Since, when a cube barely begins to move, the impulse is equal to the friction on its base, and the frictions under the cubes (when motion is impending) are proportional to their volumes (see above) ∴ we have

$$\frac{P_m}{P_n} = \frac{x_m^3}{x_n^3} \quad \ldots(1)$$

Also, the impulses on the cubes, whatever the velocity, are proportional to the face areas and to the squares of the velocities (nearly) (§ 521 eq. 1) ∴

$$\frac{P_m}{P_n} = \frac{v_m^2\, x_m^2}{v_n^2\, x_n^2} \quad \ldots(2)$$

From (1) and (2) we have $\frac{v_m^2}{v_n^2} = \frac{x_m}{x_n}$ ..., i.e. $\frac{x_m^2}{x_n^2} = \frac{v_m^4}{v_n^4}$ which substituted in eq. (2) gives

$$P_m : P_n :: v_m^6 : v_n^6 \quad \text{Q.E.D.} \quad (3)$$

Thus we see in a general way why it is that if the velocity of a stream is doubled its transporting power is increased about sixty-four fold; i.e., it can now impel along the bottom pebbles sixty-four times as heavy as the heaviest which it could move before (of same shape and substance).

Though rocks are generally from 2 to 3 times as heavy as water their loss of weight under water causes them to encounter less friction on the bottom than if not immersed. From Du Buat's experiments it appears that a velocity:

of at least $\frac{1}{4}$ ft. per sec. is necessary to transport silt;
" " $\frac{1}{2}$ " " " " " loam;
" " 1.0 " " " " " sand;
2 ft p. sec .... gravel; $3\frac{1}{2}$ ft. p. sec. — pebbles an inch in diameter;
4 ft, broken stone; 5 " " chalk, soft shale.